Microeconomics I – Classes Wrap-up

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1 Class 01

1.1 Introduction – The Consumer Problem

The standard representation of the consumer problem is the maximization of an utility function:

$$\max_{\{x_1,\ldots,x_n\}} \quad u_i(x_1,\ldots,x_n) \quad \text{s.t.} \quad \sum_{i=1}^{n} p_i x_i = y \quad ; \quad n \,\text{goods}.$$

The typical way to solve this is using the Lagrangian

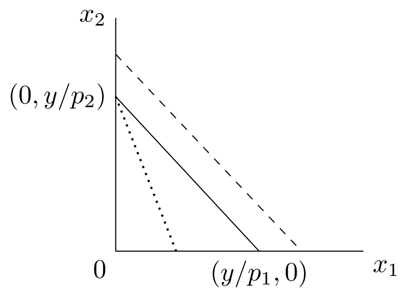
$$\mathcal{L} = u(x_1,\ldots,x_n) + \lambda(y - \sum_{i=1}^{n} p_i x_i),$$

and the first-order conditions are

- $u_i = \frac{\partial u}{\partial x_i} = \lambda p_i$, n equations;
- $y = \sum_{i=1}^{n} p_i x_i$, 1 equation.

The solution to this classical problem is the Marshallian demand, $x_i = f^i(\mathbf{p}, y), i = 1,\ldots,n$, $\mathbf{p} = 1,\ldots,n$.

1.2 The Budget Constraint (DM, cap. 1)

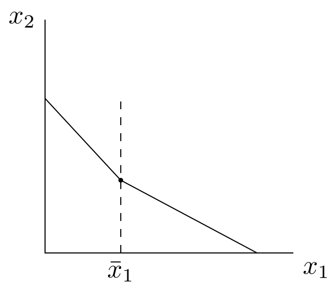


- The solid line represents the traditional linear budget constraint;
  - The slope of it is given by $-p_1/p_2$.
- The dashed line is parallel to the original budget constraint, and stands for $\Delta y > 0$;
- The dotted line is a "rotation" (in $(0, y/p_2)$) of the original budget constraint, and stands for $\Delta p_1 > 0$.

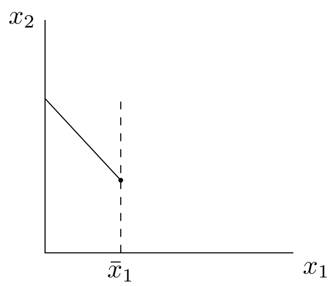
[†]These are my own notes on the first microeconomics course of Insper's PhD in Economics, which was taught by Rodrigo Soares in 2022; in this particular period, the course was assisted by Stéphanie Shinoki. I wrote it to have a better grasp on the concepts, and help others to achieve the same. Throughout the document, I'll refer to the basic bibliography as prof Soares does, that is, DM stands for Deaton and Muellbauer (1980), JR stands for Jehle and Reny (2001), MWG stands for Mas-Collel et al (1995), and TR stands for Tirole (1988). If you find anything wrong or inconsistent, please let me know: heitoraol@al.insper.edu.br.

Generally, we consider linear budget constraints (BCs), but DM discusses cases of non-linear BCs, such as:

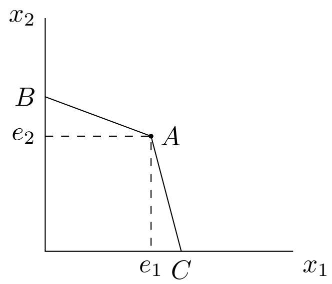
- Two-part tariff: discounts on quantity. Let $p^1 > p^0$. If $x_1 < \bar{x}_1$, then $p = p^0$; otherwise, if $x_1 \geq \bar{x}_1$, then $p = p^1$;



- Quotas: limitation on consumption. Maximum amount of x_1 is $\bar{x}_1$;



- Endowment economy: income is given by the value of what agents already have. In this case, $y = \sum_{i=1}^{n}(p_i \cdot e_i^j)$, where e_i^j is the endowment of agent j of the good i.
  - Suppose that sale prices are lower than purchase prices, and that the endowments are given by e_1 and e_2. Then, graphically[1], we get that:



Back to the case of linear BCs, suppose that Marshallian demand functions[2] does exist, $x_i = f^i(\mathbf{p}, y)$, $i = 1, \ldots, n$, and that there's no monetary illusion. Also, suppose that individuals always spend all their income, i.e., $\sum(p_i \cdot x_i) = y$. Then, the adding-up property, or Walras' Law, holds:

$$\sum_{i=1}^{n} p_i \cdot f^i(\mathbf{p}, y) = y. \tag{1}$$

[1]Think this graph in MRS terms. The segment AB is almost horizontal. Notice that if we start at B, a small $\Delta x_2 < 0$ generates a big $\Delta x_1 > 0$. That's why we say it's the "buyer of x_1" segment. Similarly, segment AC is the "buyer of x_2" segment, once a small $\Delta x_1 < 0$ generates a big $\Delta x_2 > 0$.

[2]Prof. Rodrigo exchanges f^i and x_i when writing about elasticity and implications.

2 Class 02

2.1 The Budget Constraint (Cont'd)

The adding-up property tells us that if price p_j changes, consumption as a whole also must adapt to accommodate this change, once income y has not changed:

$$\sum_{i=1}^{n} p_i \cdot f^i(\mathbf{p}, y) = y \overset{d/dp_j}{\Leftrightarrow} \underbrace{p_j \cdot \frac{\partial f^j(\mathbf{p}, y)}{\partial p_j}}_{x_j} + \sum_{i \neq j}^{n} p_i \cdot \underbrace{\frac{\partial f^i(\mathbf{p}, y)}{\partial p_j}}_{(*)} = 0, \tag{2}$$

where (*) is the variation in demand (Δf^i) of each good i given Δp_j, $i \neq j$. Notice that expenditure as a whole doesn't change. This relation says that, given the new expenditure with good j (x_j), changes in demands of all other goods must be such that the total variation in consumption, along with the new demand x_j, is zero.

Homogeneity. Marshallian demand functions are homogeneous of degree zero, i.e.,

$$f^i(\theta \mathbf{p}, \theta y) = \theta^0 \cdot f^i(\mathbf{p}, y) = f^i(\mathbf{p}, y).$$

Thus, any relabeling of prices (like multiplying everything by θ) doesn't change total consumption, that is, there's no monetary illusion.

Elasticities. They come from comparative statics over the Marshallian demand functions, and are given as follows:

$$\begin{aligned} \eta_i &\equiv \frac{\partial ln f^i(\mathbf{p}, y)}{\partial ln(y)} = \frac{\partial f^i(\mathbf{p}, y)}{\partial y} \cdot \frac{y}{f^i(\mathbf{p}, y)} \\ \varepsilon_{ij} &\equiv \frac{\partial ln f^i(\mathbf{p}, y)}{\partial ln(p_j)} = \frac{\partial f^i(\mathbf{p}, y)}{\partial p_j} \cdot \frac{p_j}{f^i(\mathbf{p}, y)}, \end{aligned} \tag{3}$$

where η_i is the income-elasticity of demand for x_i given a Δy, and ε_{ij} is the price-elasticity of demand for x_i given a Δp_j. With these identities in (3), we can rewrite the adding-up property as follows:

$$\sum_{i=1}^{n} p_i \cdot f^i(\mathbf{p}, y) = y \overset{d/dy}{\Leftrightarrow} \sum_i p_i \cdot \frac{\partial f^i}{\partial y} = 1 \overset{(\frac{x_i y}{x_i y}=1)}{\Rightarrow} \sum_i \underbrace{\frac{p_i x_i}{y}}_{S_i} \cdot \underbrace{\frac{\partial f^i}{\partial y} \cdot \frac{y}{x_i}}_{\eta_i} = 1 \Rightarrow \sum_i S_i \cdot \eta_i = 1, \tag{4}$$

where $S_i = \frac{p_i x_i}{y}$ is the share of income spent on x_i, which implies that $\sum_{i=1}^{n} S_i = 1$. Now, with (4), we can rewrite (2) as:

$$\begin{aligned} x_j + \sum_i p_i \cdot \frac{\partial f^i}{\partial p_j} &= 0 \overset{(p_j/y)}{\Rightarrow} \\ \underbrace{\frac{x_j p_j}{y}}_{S_j} + \sum_i p_i \cdot \frac{\partial f^i}{\partial p_j} \cdot \frac{p_j}{y} &= 0 \overset{(\frac{x_i}{x_i}=1)}{\Rightarrow} \\ S_j + \sum_i \underbrace{\frac{p_i x_i}{y}}_{S_i} \cdot \underbrace{\frac{\partial f^i}{\partial p_j} \cdot \frac{p_j}{x_i}}_{\varepsilon_{ij}} &= 0 \Leftrightarrow \\ S_j + \sum_i S_i \cdot \varepsilon_{ij} &= 0. \end{aligned} \tag{5}$$

The final equation in (5) is called Cournot Aggregation. Further, we can also rewrite the homogeneity property from Euler's Theorem[3] as:

$$
\begin{aligned}
f^i(\mathbf{p}, y) \overset{(*)}{\Rightarrow} \frac{\partial f^i}{\partial y} \cdot y + \sum_{j=1}^{n} \frac{\partial f^i}{\partial p_j} \cdot p_j = 0 \overset{(1/x_i)}{\Rightarrow} \\
\underbrace{\frac{\partial f^i}{\partial y} \cdot \frac{y}{x_i}}_{\eta_i} + \sum_{j=1}^{n} \underbrace{\frac{\partial f^i}{\partial p_j} \cdot \frac{p_j}{x_i}}_{\varepsilon_{ij}} = 0 \Leftrightarrow \\
\eta_i + \sum_{j=1}^{n} \varepsilon_{ij} = 0,
\end{aligned} \tag{6}
$$

where in (*) we use the fact that the Marshallian demand function $f^i(\mathbf{p}, y)$ is homogeneous of degree zero, and thus the sum of its partial derivatives w.r.t. each argument (all prices in the vector $\mathbf{p}$, and y) equals zero. Thus, from the final equation in (6), we have that the homogeneity property implies that all demand elasticities of a good sum up to zero.

Moreover, we have some market forces implicit in the BC:

- Assume consumers act randomly, but spend all of their income;
- Uniformly distributed over the BC;
- Average demand in this economy is $x_1 = \frac{y}{2p_1}$, $x_2 = \frac{y}{2p_2}$.

2.2 Preferences and Utility (JR, cap. 1)

2.2.1 Preference Relations and Axioms

First, we define a consumption set X, which is the set of all conceivable consumption options, whether they are affordable or not. We make the following assumptions over X:

- $\varnothing \neq X \subseteq \mathbb{R}^n_+$;
- X is closed;
- X is convex (all convex combinations of $\mathbf{x}^1, \mathbf{x}^2 \in X$ are in X);
- $\mathbf{0} \in X$.

Further, for any $\mathbf{x}^1, \mathbf{x}^2 \in X$, we define preference relations as follows:

- $\succsim$: if $\mathbf{x}^1 \succsim \mathbf{x}^2$, we say that "$\mathbf{x}^1$ is at least as good as $\mathbf{x}^2$";
- $\succ$: if $\mathbf{x}^1 \succsim \mathbf{x}^2$, and it's not true that $\mathbf{x}^2 \succsim \mathbf{x}^1$, we write $\mathbf{x}^1 \succ \mathbf{x}^2$, and say "$\mathbf{x}^1$ is strictly preferred to $\mathbf{x}^2$";
- $\sim$: if $\mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^1$, we write $\mathbf{x}^1 \sim \mathbf{x}^2$, and say that "$\mathbf{x}^1$ is indifferent to $\mathbf{x}^2$".

[3]Euler's Theorem for homogeneous functions: if f is a function of n real variables that is positively homogeneous of degree k, and it is continuously differentiable in some open subset of $\mathbb{R}^n$, then it satisfies the partial differential equation in this open set

$$\sum_{i=1}^{n} x_i \cdot \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} = k \cdot f(x_1, \ldots, x_n).$$

Based on this, we can define sets in relation to a given bundle $\mathbf{x}^0 \in X$:

$$\succsim (\mathbf{x}^0) \equiv \{\mathbf{x} \mid \mathbf{x} \in X,\ \mathbf{x} \succsim \mathbf{x}^0\},$$

where $\mathbf{x}$ and $\mathbf{x}^0$ are vectors of n goods. Obviously, we can define similar sets for $\precsim (\mathbf{x}^0)$, $\succ (\mathbf{x}^0)$, $\prec (\mathbf{x}^0)$, $\sim (\mathbf{x}^0)$.

Since not all conceivable consumption option is actually affordable, we define the budget set B, which contains all feasible consumption options for the individual. The budget set is defined as

$$\mathbb{B} \equiv \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n_+,\ \mathbf{p} \cdot \mathbf{x} \leq y\},$$

where $\mathbf{x}$ is the vector of n goods, and $\mathbf{p}$ is the vector of prices for those n goods. With this setting, we can define the axioms on preferences.

Axiom 1: Completeness. *For all* $\mathbf{x}^1, \mathbf{x}^2 \in X$, $[\mathbf{x}^1 \succsim \mathbf{x}^2] \vee [\mathbf{x}^1 \succsim \mathbf{x}^2] \vee [\mathbf{x}^1 \sim \mathbf{x}^2]$. It means that individual can compare any pair of bundles.

Axiom 2: Transitivity. *For all* $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in X$, *if* $[\mathbf{x}^1 \succsim \mathbf{x}^2] \wedge [\mathbf{x}^2 \succsim \mathbf{x}^3]$, *then* $\mathbf{x}^1 \succsim \mathbf{x}^3$. This is the main axiom, and says that the individual is able to order preferences. Along Axiom 1, it allows the existence of the preference relation. When this axiom fails (e.g., Dutch Book), we say that the preference is not consistent.

Axiom 3: Continuity. *For all* $\mathbf{x} \in \mathbb{R}^n_+$, $\succsim (\mathbf{x})$ *is closed in* $\mathbb{R}^n_+$. From JR, it's also possible to write this axiom as "both $\succsim (\mathbf{x})$ and $\precsim (\mathbf{x})$ are closed in $\mathbb{R}^n_+$". The main implication of this axiom is over utility functions, since it rules out the possibility of open areas (like gaps) in the indifference set (or indifference curves).

Axiom 4': Local non-satiation. *For all* $\mathbf{x}^0 \in \mathbb{R}^n_+$, *and for all* $\varepsilon > 0$, *there exists some* $\mathbf{x} \in B_\varepsilon(\mathbf{x}^0) \cap \mathbb{R}^n_+$ *such that* $\mathbf{x} \succ \mathbf{x}^0$, *where* $B_\varepsilon(\mathbf{x}^0)$ *is the neighborhood of* $\mathbf{x}^0$. Its main implication is over utility functions, since it rules out the possibility of "zones of indifference" (balls) within the indifference set. We can also write this axiom as

$$\forall \mathbf{x}^0 \in \mathbb{R}^n_+,\ \forall \varepsilon > 0,\ \exists \mathbf{x} \in \mathbb{R}^n_+ \quad s.t. \quad [||\mathbf{x} - \mathbf{x}^0|| \leq \varepsilon] \wedge [\mathbf{x} \succ \mathbf{x}^0].$$

Axiom 4: Strict monotonicity. *For all* $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, *if* $\mathbf{x}^0 \geq \mathbf{x}^1$, *then* $\mathbf{x}^0 \succsim \mathbf{x}^1$, *while if* $\mathbf{x}^0 \gg \mathbf{x}^1$, *then* $\mathbf{x}^0 \succ \mathbf{x}^1$. Axiom 4 is a stronger version of Axiom 4', and it says that if $\mathbf{x}^0$ contains at least the same amount of every good as $\mathbf{x}^1$, then $\mathbf{x}^0 \succsim \mathbf{x}^1$; if it contains more of every good, then $\mathbf{x}^0 \succ \mathbf{x}^1$. Its main implication is that, besides implying Axiom 4', it eliminates the possibility that the indifference sets (or curves) in $\mathbb{R}^n_+$ "bend upward", or contain positively sloped segments. It also requires that the "preferred to" sets to be "above" the indiff sets, while the "worse than" sets must be "below" them.

Axiom 5': Convexity. *If* $\mathbf{x}^1 \succsim \mathbf{x}^0 \Rightarrow t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$, $\forall t \in [0,1]$. This axiom says that convex combinations of bundles are at least as good as the least preferred bundle. Notice that it allows flat portions[4] in the indifference curve of utility functions.

[4] Thus, linear utility functions are convex.

Axiom 5: Strict convexity. *If* $\mathbf{x}^1 \neq \mathbf{x}^0$, *and* $\mathbf{x}^1 \succsim \mathbf{x}^0$, *then* $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succ \mathbf{x}^0$, $\forall t \in (0,1)$. Axiom 5 is a stronger version of 5', and states that convex combinations of bundles are strictly better than the least preferred bundle; it's the idea of "weighted averages better than extremums". The key difference of Axiom 5 in relation to 5' is in the set where t is defined. Axiom 5' defines it in the closed set $[0,1]$, i.e., t could be 0 or 1, allowing situations where only $\mathbf{x}^1$ or only $\mathbf{x}^0$ are consumed; axiom 5 defines t in the open set $(0,1)$, i.e., t provides only (and all) situations where both bundles (via convex combinations) are consumed. Moreover, it rules out the possibility of flat portions in the indifference curve of utility functions, generating the famous convex form of indifference curves we know.

Generally, we say that preferences are "well-behaved" when axioms 1-5 are satisfied. The classical Cobb-Douglas utility function is an example of well-behaved preference.

3 Class 03

3.1 Preferences and Utility (cont'd)

3.1.1 The Utility Function

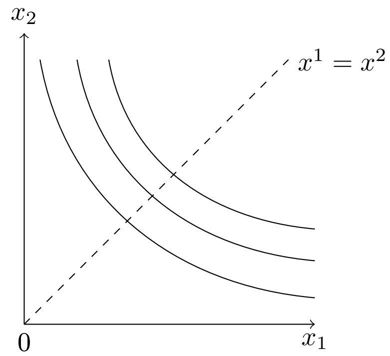
Definition. We call $u : \mathbb{R}^n_+ \to \mathbb{R}$ a utility function representing preferences $\succsim$ if,

$$\forall \mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+,\ u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \Leftrightarrow \mathbf{x}^0 \succsim \mathbf{x}^1.$$

If the binary relation $\succsim$ is complete, transitive, continuous, and strictly monotone[5], there exists a real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ that represents $\succsim$[6].

Also, if $u(\cdot)$ represents $\succsim$, and there exists some f such that $v(\cdot) = f(u(\cdot))$, where $f(\cdot)$ is strictly increasing in $u(\cdot)$, then $v(\cdot)$ also represents the same preferences $\succsim$. This means that the utility function is unique up to positive monotone transformations. Moreover,

If $u(\cdot)$ is strictly increasing[7] $\Leftrightarrow$ $\succsim$ is strictly monotone;
If $u(\cdot)$ is (strictly) quasiconcave[8] $\Leftrightarrow$ $\succsim$ is (strictly) convex.



This indifference curves respect all axioms.

[5]Monotonicity overall is not strictly necessary.

[6]The full proof of the existence of the utility function is available at JR, p. 14-16. It's worth looking.

[7]This implies that the individual prefers more than less of each good.

[8]This implies that the indifference curves are (strictly) convex w.r.t. the origin.

4 Class 04

4.1 Preferences and Utility (cont'd)

4.1.1 The Consumer's Problem

General formulation: $\mathbf{x}^* \in \mathbb{B}$ s.t. $\mathbf{x}^* \succsim \mathbf{x}$, $\forall \mathbf{x} \in \mathbb{B}$, where $\mathbb{B}$ is the set of feasible consumption options. Thus, $\mathbf{x}^*$ is the best consumption option among all feasible ones, and it is what the consumer searches for solving his problem. We typically write the consumer's problem as

$$\max_{\mathbf{x} \in \mathbb{R}^n_+} \quad u(\mathbf{x}) \qquad s.t. \qquad \mathbf{p}'\mathbf{x} \leq y, \tag{7}$$

where $\mathbf{x}$ is the vector of n goods, and $\mathbf{p}'$ is the transposed vector of n prices, i.e., $\mathbf{p}'\mathbf{x} = \sum_{i=1}^n p_i x_i$. From the Weierstrass Theorem:

- If $u(\cdot)$ is continuous[9], and $\mathbb{B}$ is a compact set, then the maximum of $u(\cdot)$ exists;
- If $u(\cdot)$ is strictly increasing and quasiconcave, then the solution is unique, and it's on the budget line.

With differentiability, we can use the Langrange method to solve equation (7):

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda(y - \mathbf{p}'\mathbf{x}),$$

where λ is the *Lagrange multiplier.* The solution can be characterized by the Kuhn-Tucker (first-order) conditions (FOCs) of this problem:

$$\begin{aligned}
&\text{(i) } \frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u(x)}{\partial x_i} - \lambda \cdot p_i \leq 0 \quad \text{(will be an equality if } x_i > 0) \\
&\text{(ii) } y - \mathbf{p}'\mathbf{x} \geq 0 \\
&\text{(iii) } \lambda \cdot (y - \mathbf{p}'\mathbf{x}) = 0
\end{aligned}$$

In (i), λ is called marginal value of income[10], and it is so because it "transforms" p_i from monetary units into utility units (or whatever units the objective function works with); and it is an inequality because, a priori, there's no reason to be sure that $x_i^* > 0$; and if the good is not purchased, then it doesn't matter how its price affect income through λ (which is the same as saying that $\lambda = 0$, or that the marginal utility of $x_i < 0$). Also, it will not be the case that (ii) is an equality, because since there are commodities not purchased, then not all income is spent for achieving maximum utility, and then (ii) is an inequality. Moreover, if some $x_i^* = 0$, then in (iii), we have that the marginal value of income for that good (λ_i) is zero, since not all income in the world would make utility maximum, once not all commodities are purchased.

On the other hand, if all $x_i^* > 0$, then all restrictions are satisfied with equality. In (i), the intuition is that income is actually crucial for achieving maximum utility (this is implied by monotonicity); thus for each price p_i, λ_i will be positive, such that λ_i transforms p_i to be equal to the marginal utility of

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[9] Assuming that $u(\cdot)$ is continuous and strictly monotone, we can say that

(i) $x = f(\mathbf{p}, y)$, which is the Marshallian demand function, is homogeneous of degree zero;

(ii) Walras' law holds: $\mathbf{p}'\mathbf{x} = y$.

[10] From the Envelope Theorem:

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial u(x(\mathbf{p}, y))}{\partial y} = \lambda.$$

that good (i.e., the price I pay for the good is exactly how much my utility function says the good is worth). In (ii), the intuition is that all income will be used to purchase the goods, once that's exactly what makes utility higher, i.e., spend money in purchase of goods. In (iii), since there is marginal utility in income w.r.t. all goods ($\lambda_i > 0, \forall i$), then we will have that $y - \mathbf{p}'\mathbf{x} = 0$, and we will be able to write (ii) as $y = \mathbf{p}'\mathbf{x}$.

Hence, for well-behaved functions, which is the general case, $u(\cdot)$ is monotonically increasing (Axiom 4), and thus the constraints are satisfied with equalities, since then λ matters ($\lambda > 0$). Then, we consider the FOCs as

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u(x_i)}{\partial x_i} - \lambda p_i = 0 \text{ (with } x_i > 0),$$
$$\mathbf{p}'\mathbf{x} = y \text{ (with } \lambda > 0).$$

Beyond having constraints satisfied with equalities, if $u(\cdot)$ is also strictly quasiconcave (Axiom 5), which will be the case if preferences are well-behaved, then its level curves (= indifference curves) will be strictly quasiconvex, and thus the solution to the utility maximization problem in (7) is unique, and it will be at the tangency of the objective function $u(\cdot)$ and the constraint function (budget set).

Considering the two-goods case for well-behaved preferences, the FOCs can be written as

$$[x_i] : \frac{\partial u(x_i^*, x_j^*)}{\partial x_i} = \lambda p_i$$
$$[x_j] : \frac{\partial u(x_i^*, x_j^*)}{\partial x_j} = \lambda p_j$$
$$[\lambda] : p_i x_i^* + p_j x_j^* = y.$$

Solving the FOCs leads to the Marshallian demand functions. Notice that FOCs are necessary but not sufficient to guarantee that the solution to the FOCs is actually the demand that maximizes utility. The second-order conditions[11] check if the bundle found with Lagrange method is indeed a maximum for the maximization problem. But, if preferences are well-behaved, then the SOCs are satisfied straightforwardly.

Dividing the FOC in x_i by x_j gives us that, if preferences are well-behaved, consumer's optimum choice satisfies the tangency condition in which the absolute value of the MRS is equal to the price ratio.

$$MRS_{ij} = \frac{\partial u(x_i^*, x_j^*)/\partial x_i}{\partial u(x_i^*, x_j^*)/\partial x_j} = \frac{p_i}{p_j}$$

Thus, if preferences are well-behaved, the solution will be interior, i.e., both goods are consumed in positive amounts ($x_i^* > 0, x_j^* > 0$). Axiom 5 implies that the absolute value of the MRS_{ij} decreases along the indifference curve, i.e., for a given utility level, the value of good i, in terms of good j, decreases the more good i the individual possesses.

4.1.2 More on Kuhn-Tucker Conditions

For the full intuition, see JR (2011, p. 595-597) and MWG (1995, p. 959-962).

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[11] SOCs are obtained from the bordered Hessian matrix of Lagrange function, by computing the Lagrangian's second-order partial derivatives. This matrix must be negative definite in order to secure the demand functions computed by the FOCs are really the optimum demands for individual analyzed. For the two-goods case, we can show that SOCs are satisfied if the following inequality holds:

$$2u_i u_j u_{ji} - u_j^2 u_{ii} - u_i^2 u_{jj} > 0,$$

where the functions above are evaluated at the optimum candidate.

The Kuhn-Tucker (KT) conditions are first-order conditions that provide a unified treatment of constrained optimization problems:

- Allowing for inequality constraints;
- Allowing for any number of constraints;
- Constraints may be binding or not at the solution;
- Allowing for non-negative constraints ($x_i \geq 0, \forall i$);
- Allowing for boundary (corner) solutions ($x_i = 0$ for some i);
- Using dual variables (Lagrange multipliers $\mathcal{L}$) as shadow variables (marginal values).

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable, and tht $\mathbf{b} \in \mathbb{R}^m$. Let (P) be a maximization problem such that

$$\max_{\mathbf{x} \in \mathbb{R}^n} \quad f(\mathbf{x}) \qquad s.t. \qquad \begin{cases} \mathbf{x} \geq 0 \\ G(\mathbf{x}) \leq \mathbf{b} \end{cases}$$

and let $(*)$ mean that $\hat{\mathbf{x}}$ is a solution of the problem (P).

Let G^i be the i-th constraint of (P). The Kuhn-Tucker conditions are as follows:

$$\begin{aligned} &\exists \lambda_1, \ldots, \lambda_m \in \mathbb{R}_+ \text{ such that (at } \hat{\mathbf{x}}\text{):} \\ &\quad \text{(i) for } j = 1, \ldots, n : \frac{\partial f}{\partial x_j} \leq \sum_{i=1}^{m} \lambda \cdot \frac{\partial G^i}{\partial x_j} \text{ (and = if } \hat{x}_j > 0); \\ &\quad \text{(ii) for } i = 1, \ldots, m : G^i(\hat{\mathbf{x}}_i) \leq b_i \text{ (and = if } \lambda_i > 0). \end{aligned}$$

In vectorial form, we have that

$$\begin{aligned} &\exists \lambda \in \mathbb{R}_+^m \text{ such that (at } \hat{\mathbf{x}}\text{):} \\ &\quad \text{(i) } [\nabla f \leq \sum_{i=1}^{m} \lambda_i \nabla G^i] \wedge [\hat{\mathbf{x}} \cdot (\nabla f - \sum_{i=1}^{m} \lambda_i \nabla G^i) = 0]; \\ &\quad \text{(ii) } [G(\hat{\mathbf{x}}) \leq \mathbf{b}] \wedge [\lambda \cdot (G(\hat{\mathbf{x}}) - \mathbf{b}) = 0]. \end{aligned}$$

Condition (i) states that the partial derivative of the objective function w.r.t. all variables of choice must be no greater than the sum of linear combinations of some scalar (λ) with the partial derivative of the constraints w.r.t. to those variables. In vectorial terms, condition (i) tells us that the gradient of the objective function must be within the cone (region of linear combinations) generated by the gradients (times λ) of the constraints. Also, we can say that this condition states that, at the solution point ($\hat{\mathbf{x}}$), all constraints are linearly independent, which implies that ∇f must be a non-negative linear combination of the linearly independent set of gradients of the constraint functions. In words, it says that all λ_i are non-negative scalars that transforms each gradient of the constraint functions in such a way that, when summed up, they are equal or greater than the gradient of the objective function; all of this at the solution point $\hat{\mathbf{x}}$.

Condition (ii) is called complementary slackness (JR, p. 600). It says that if a constraint is slack (non-binding) its associated Lagrange multiplier must be zero (and then no linear transformation would make the constraint function useful to attend (i)); otherwise, if a Lagrange multiplier is positive, then its associated constraint must be binding, i.e., at the optimal point $\hat{\mathbf{x}}$, the constraint is satisfied with equality. The Lagrange multiplier λ_j can be interpreted as the marginal increase in the objective function when the j-th constraint is relaxed.

As for the second-order conditions (SOCs), we have that they're related to the curvature of the objective function and the constraint functions. Briefly, we need the bordered Hessian matrix of the Lagrange function to be negative definite. Using our economic notation, and considering the two-goods case, the FOCs are

$$[x_i] : \frac{\partial u(x_i^*, x_j^*)}{\partial x_i} - \lambda p_i = 0; \qquad [x_j] : \frac{\partial u(x_i^*, x_j^*)}{\partial x_j} - \lambda p_j = 0; \qquad [\lambda] : p_i x_i^* + p_j x_j^* = y.$$

The SOCs for this case are applied on the left-hand-side of $[x_i]$ and $[x_j]$. If it was an unconstrained optimization problem, then the SOCs would be

$$\frac{\partial^2 u}{\partial x_i^2} < 0; \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}; \qquad \frac{\partial^2 u}{\partial x_j^2} < 0 \Leftrightarrow$$

$$\Leftrightarrow H \equiv \begin{bmatrix} f_{x_i x_i} & f_{x_i x_j} \\ f_{x_j x_i} & f_{x_j x_j} \end{bmatrix}$$

That's the Hessian matrix H of the Lagragian function (or just the utility function, once we're concerned only about f in the unconstrained case). In this case, if the first term in the upper left corner is negative ($f_{x_i x_i} < 0$), then the solution is indeed a maximum. But for the constrained optimization case, we need to compute also the second derivatives of $[\lambda]$ w.r.t. to x_i and x_j, which gives the bordered Hessian matrix ($\bar{H}$) of the Lagrangian function

$$\bar{H} \equiv \begin{bmatrix} 0 & \mathcal{L}_{\lambda x_i} & \mathcal{L}_{\lambda x_j} \\ \mathcal{L}_{\lambda x_i} & f_{x_i x_i} & f_{x_i x_j} \\ \mathcal{L}_{\lambda x_j} & f_{x_j x_i} & f_{x_j x_j} \end{bmatrix}$$

Notice first that it's a symmetric matrix, which is a requirement. Further, to secure that the candidate point is indeed a maximizer of the objective function, $\bar{H}$ must be negative definite, which in this case of a bordered Hessian matrix means that the determinant of $\bar{H}$ must be positive. This is the same as saying that, evaluated at the optimum candidate $\hat{\mathbf{x}}$, the following inequality is satisfied

$$(\mathcal{L}_{\lambda x_i} \cdot \mathcal{L}_{x_j x_i} \cdot \mathcal{L}_{\lambda x_j} + \mathcal{L}_{x_i x_j} \cdot \mathcal{L}_{\lambda x_i} \cdot \mathcal{L}_{\lambda x_j}) - (\mathcal{L}_{\lambda x_j} \cdot \mathcal{L}_{x_i x_i} \cdot \mathcal{L}_{\lambda x_j} + \mathcal{L}_{\lambda x_i} \cdot \mathcal{L}_{\lambda x_i} \cdot \mathcal{L}_{x_j x_j}) > 0 \Leftrightarrow$$

$$\Leftrightarrow 2 \cdot \mathcal{L}_{\lambda x_i} \cdot \mathcal{L}_{x_j x_i} \cdot \mathcal{L}_{\lambda x_j} - \mathcal{L}^2_{\lambda x_j} \cdot \mathcal{L}_{x_i x_i} - \mathcal{L}^2_{\lambda x_i} \cdot \mathcal{L}_{x_j x_j} > 0.$$

But as we've noticed, for what matters in microeconomics, the SOCs will always be satisfied for well-behaved utility preferences, so there's no need to check them if the preference of interest satisfies Axioms 1 to 5.

Finally, we have the Kuhn-Tucker theorems. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}^m$ are both continuously differentiable, and that $\mathbf{b} \in \mathbb{R}^m$. Also, let (a), (b'), and (b") be the SOCs.

Theorem: KT $\Rightarrow$ $(*)$, i.e., KT is a sufficient condition for $\hat{\mathbf{x}}$ to be a solution of (P), if

(a) each G^i is quasiconvex, and

(b') f is concave, or

(b") f is quaseiconcave and $\nabla f \neq \mathbf{0}$ at $\hat{\mathbf{x}}$.

Theorem: $(*)$ $\Rightarrow$ KT, i.e., KT is a necessary condition for $\hat{\mathbf{x}}$ to be a solution of (P), if

(a) f is quasiconcave, and

(b') each G^i is quasiconvex, and

(b") the constraint set $\{\mathbf{x} \in \mathbb{R}^n \mid G(\hat{\mathbf{x}}) \leq \mathbf{b}\}$ satisfies a constraint qualification.

In the first version, the theorem states that, given that (a) and (b') or (b") holds, if all KT conditions are satisfied at a particular point $\hat{\mathbf{x}}$, then $\hat{\mathbf{x}}$ is indeed a solution to (P). In the second version, the theorem states that, given that (a) and (b') or (b") holds, for a particular point $\hat{\mathbf{x}}$ to be a solution to (P), it is necessary but not sufficient that all KT conditions are satisfied at $\hat{\mathbf{x}}$. Jointly, both versions of the KT theorems tells us that every point that is a solution to (P) satisfies KT conditions, but not every point that satisfies KT conditions is a solution to (P).

4.2 Marshallian demands and the Indirect Utility Function

Marshallian demands are the solution to the FOCs, and arise from solving the consumer's utility maximization problem. We denote them as

$$x_i = f_i(\mathbf{p}, y), \quad y = 1, \ldots, n.$$

The Indirect Utility Function is the utility function $u(\cdot)$ in terms of prices and income only.

$$V(\mathbf{p}, y) \equiv u(f(\mathbf{p}, y)) = \max_{\mathbf{x}} \{u(\mathbf{x}) \ \ s.t. \ \ \mathbf{p}'\mathbf{x} = y\}, \ f(\mathbf{p}, y) = [f_1(\mathbf{p}, y), \ldots, f_n(\mathbf{p}, y)].$$

It provides the maximum utility that $u(\cdot)$ gives under the budget constraint of the maximization problem. It is obtained by using the Marshallian demands back into the utility function $u(\cdot)$. It has the following properties:

- Continuity (from "maximum theorem");
- Homogeneous of degree zero on $(\mathbf{p}, y)$, once Marshallian demands are also homogeneous of degree zero on $(\mathbf{p}, y)$ (no monetary illusion):

$$V(t\mathbf{p}, ty) = \max_{\mathbf{x}} \{u(\mathbf{x}) \ \ s.t. \ \ t\mathbf{p}'\mathbf{x} = ty\} = \max_{\mathbf{x}} \{u(\mathbf{x}) \ \ s.t. \ \ \mathbf{p}'\mathbf{x} = y\}$$

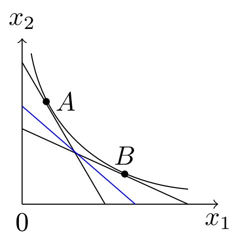
- Strictly increasing on y (from Envelope Theorem):

$$\frac{\partial V(\mathbf{p}, y)}{\partial y} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial y} = \lambda^* > 0$$

- Decreasing in p:

$$\frac{\partial V(\mathbf{p}, y)}{\partial p_i} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial p_i} = -\lambda x_i \leq 0$$

- Quasiconvex on $(\mathbf{p}, y)$, which means that individual prefers more extreme combinations of prices and income (i.e., more extreme utility) rather than convex (intermediary) combinations. In the graph below, the blue line is an intermediary combination of those budget sets. But we note that the quasi-convexity imply that this medium solution is worse than A or B. That's why the individual prefers extreme combinations, because of the substitutability of the goods.



- Roy's Identity: we can recover the Marshallian demand from the indirect utility function by doing

$$f_i(\mathbf{p}, y) = -\frac{\partial V / \partial p_i}{\partial V / \partial y}, \quad i = 1, \ldots, n,$$

which holds from the Envelope Theorem:

$$\frac{\partial V}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = -\lambda x_i^* = -\frac{\partial V}{\partial y} \cdot x_i^* \Rightarrow x_i^* = -\frac{\partial V/\partial p_i}{\partial V/\partial y}.$$

Further, the following relations holds:

Individual (strictly) prefers weighted average bundles rather than extreme ones $\Leftrightarrow$ $\Leftrightarrow \succsim$ is (strictly) convex $\Leftrightarrow u(\cdot)$ is (strictly) quasiconcave $\Leftrightarrow$ indifference curves are (strictly) convex.
Individual (strictly) prefers extreme bundles rather than weighted average ones $\Leftrightarrow$ $\Leftrightarrow \succsim$ is (strictly) concave $\Leftrightarrow u(\cdot)$ is (strictly) quasiconvex $\Leftrightarrow$ indifference curves are (strictly) concave.

4.3 The Cost Minimization Problem

We search for the minimum cost at given prices in order to achieve a certain utility level.

$$\min_{\mathbf{x}\in\mathbb{R}^n_+} \quad \mathbf{p}'\mathbf{x} \qquad s.t. \qquad u(\mathbf{x}) = u \quad \Rightarrow \quad \mathcal{L} = \mathbf{p}'\mathbf{x} + \mu[u - u(\mathbf{x})]$$

$$(x_i): \quad p_i - \mu \cdot \frac{\partial u(\mathbf{x})}{\partial x_i} = 0, \qquad i = 1, \dots, n;$$
$$(\mu): \quad u(\mathbf{x}) = u,$$

where μ is interpreted as the marginal cost of utility. Also, from (x_i), we have that

$$MRS_{ij} = \frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{p_i}{p_j}.$$

The solution to this problem is a system of Hicksian demand functions

$$x_i = \mathbf{h}(\mathbf{p}, u) = \begin{bmatrix} h_1(\mathbf{p}, u) & h_2(\mathbf{p}, u) & \dots & h_n(\mathbf{p}, u) \end{bmatrix}, \qquad i = 1, \dots, n$$

Analogously to the Indirect Utility Function, there is the Cost (Expenditure) Function, which gives the minimal (optimal) expenditure at the given prices to achieve the utility level u:

$$e(\mathbf{p}, u) \equiv p_1 h_1(\mathbf{p}, u) + p_2 h_2(\mathbf{p}, u) + \dots + p_n h_n(\mathbf{p}, u) = \mathbf{p}'\mathbf{h}(\mathbf{p}, u) = \min_{\mathbf{x}}\{\, \mathbf{p}'\mathbf{x} \quad s.t. \quad u(\mathbf{x}) \geq u\} = e^*.$$

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5.1 The Cost Minimization Problem (cont'd)

If $u(\cdot)$ is continuous and strictly increasing (strictly monotonic), then $e(\mathbf{p}, u)$ has the following properties:

- Continuity (from "maximum theorem");
- Is strictly increasing in u:
$$\frac{\partial e(\mathbf{p}, u)}{\partial u} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \mu^*)}{\partial u} = \mu^* > 0$$
- If $u(\cdot)$ is strictly quasiconcave, we have the Shephard's lemma + increasing in p:
$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = x_i^* = h_i(\mathbf{p}, u) \geq 0,$$
since demands cannot be negative;

- Is homogeneous of degree 1 in p:

$$e(\theta\mathbf{p}, u) = \min_{\mathbf{x}} \{\theta\mathbf{p}'\mathbf{x} \;\; s.t. \;\; u(\mathbf{x}) = u\} = \theta \cdot \min_{\mathbf{x}} \{\mathbf{p}'\mathbf{x} \;\; s.t. \;\; u(\mathbf{x}) = u\} = \theta \cdot e(\mathbf{p}, u)$$

- Is concave in p.

Proof. Consider p^1, p^2 as 2 vectors of prices for the same n goods, with x^1 minimizing costs (of achieving u) at p^1, x^2 minimizing costs (of achieving u) at p^2. Define $p^t = t \cdot p^1 + (1-t) \cdot p^2$, with $t \in [0, 1]$; assume that x^* minimizes costs at p^t.

We know that

$$p^{1'} x^1 \leq p^{1'} x, \; \forall x \, \text{that achieves} \, u,$$

$$p^{2'} x^2 \leq p^{2'} x, \; \forall x \, \text{that achieves} \, u.$$

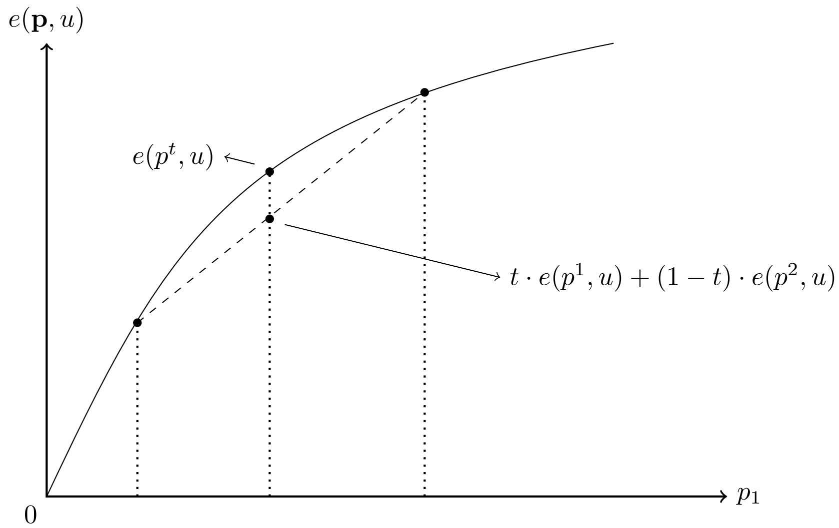
This is true in particular for x^*:

$$p^{1'} x^* \leq p^{1'} x^* \quad \wedge \quad p^{2'} x^2 \leq p^{2'} x^*.$$

Now, notice that

$$t \cdot \underbrace{p^{1'} x^1}_{e(p^1, u)} + (1-t) \cdot p^{2'} x^2 \leq [t \cdot p^{1'} + (1-t) \cdot p^{2'}] x^* = p^{t'} x^*$$

$$t \cdot e(p^1, u) + (1-t) \cdot e(p^2, u) \leq e(p^t, u).$$



□

Concavity of $e(\mathbf{p}, u)$ comes from the fact that it's possible to substitute goods given any $\Delta p_i > 0$. If p_i increases and nothing is done, costs will increase linearly (45º line in graph), just like a Leontief utility; if it's possible to substitute goods, then costs will increase less than linearly.

5.2 Price Index

Let p^1 = prices in period 1, p^0 = prices in period 0, and x^R be a reference bundle of goods. A price index is a function as the following

$$P(p^1, p^0; x^R) = \frac{p^{1\prime} \cdot x^R}{p^{0\prime} \cdot x^R}$$

which is the change in prices from period 0 to period 1, evaluated at x^R. The ideal economic price index would be the one built from the cost function

$$\frac{e(p^1, u^0)}{e(p^0, u^0)},$$

but since we cannot directly see the cost function, there are two main price indexes we use:

- Laspeyres:
$$P(p^1, p^0; x^0) = \frac{p^{1\prime} \cdot x^0}{p^{0\prime} \cdot x^0} \geq \frac{e(p^1, u^0)}{e(p^0, u^0)},$$
which superestimates increase in prices, since it considers the same bundle of "yesterday" in the current period of time, without considering substitutability given $\Delta p_i > 0$;

- Paasche:
$$P(p^1, p^0; x^1) = \frac{p^{1\prime} \cdot x^1}{p^{0\prime} \cdot x^1} \leq \frac{e(p^1, u^1)}{e(p^0, u^1)},$$
which underestimates increase in prices, since it uses that current bundle x^1 to evaluate p^0, without considering that the real bundle in period 0 could be another one, reflecting an even higher inflation during the period.

Notice that if utilities were all like Leontief, then those indexes would all be equal, since Leontief utilities don't support substitutability, i.e., $x^0 = x^1$.

5.3 Duality

Duality is a fact that emerges from analyzing both utility maximization and expenditure minimization problems. The former searches for the highest indifference curve the consumer can achieve at a given budget set, while the latter searches for the lowest budget set at a given utility level.

$\max_{\mathbf{x}} \{u(\mathbf{x}) \;\; s.t. \;\; \mathbf{p}'\mathbf{x} = y\}$	$\min_{\mathbf{x}} \{\mathbf{p}'\mathbf{x} \;\; s.t. \;\; u(\mathbf{x}) = u\}$
Sol: $x^* = f(\mathbf{p}, y)$ (Marshallian)	Sol: $x^* = h(\mathbf{p}, u)$ (Hicksian)
$f(\mathbf{p}, y)$ into $u(\mathbf{x}) \Rightarrow$ Indirect Utility Function: $V(\mathbf{p}, y)$	$h(\mathbf{p}, u)$ into $\mathbf{p}'\mathbf{x} \Rightarrow$ Cost Function: $e(\mathbf{p}, u)$

We can establish important relations between these problems:

- Inverting y in $e(\mathbf{p}, u) = y$ by u yields in $V(\mathbf{p}, y) = u$, and vice-versa;

- Substituting y in Marshallian demand $x_i = f_i(\mathbf{p}, y)$ by the cost function $e(\mathbf{p}, u)$, yields in the Hicksian demand for good i,
$$x_i = h_i(\mathbf{p}, u)$$

- Substituting u in Hicksian demand $x_i = h_i(\mathbf{p}, u)$ by the ind. util. function $V(\mathbf{p}, y)$, yields in the Marshallian demand for good i,
$$x_i = f_i(\mathbf{p}, y)$$

- By Roy's Identity, we have that
$$f_i(\mathbf{p}, y) = -\frac{\partial V(\mathbf{p}, y)/\partial p_i}{\partial V(\mathbf{p}, y)/\partial y}$$

- By Shephard's Lemma, we have that
$$h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i}$$

From these relations, the following theorems arise (see JR, p. 45):

- $f_i(\mathbf{p}, e(\mathbf{p}, u)) = f_i(\mathbf{p}, y^*) = h_i(\mathbf{p}, u)$;
- $h_i(\mathbf{p}, V(\mathbf{p}, y)) = h_i(\mathbf{p}, u^*) = f_i(\mathbf{p}, y)$;
- $V(\mathbf{p}, e(\mathbf{p}, u)) = V(\mathbf{p}, y^*) = u^*$;
- $e(\mathbf{p}, V(\mathbf{p}, y)) = e(\mathbf{p}, u^*) = y^*$,

where u^* and y^* are optimal levels of utility and income, respectively. In the first two equations, u^* is the maximum level of utility given an arbitrary level of income y, and y^* is the minimum level of income in order to reach an arbitrary level of utility u.

Further, we have the following **properties of demand**:

- Adding-up, Walras' law, or Additivity: this holds whenever the demand for goods exhausts income. It's the mathematical way of saying that the individual maximizes utility on the BC frontier line, and not under the frontier of the budget set. Formally,

$$\sum_i p_i \cdot h_i(\mathbf{p}, u) = \sum_i p_i \cdot f_i(\mathbf{p}, y) = y,$$

  which gives that $u = V(\mathbf{p}, y)$;

- Homogeneity: Hicksian demands are homogeneous of degree zero in p:

$$h_i(\mathbf{p}, u) = h_i(\theta\mathbf{p}, u).$$

  Derivatives of $e(\mathbf{p}, u)$, which is homogeneous of degree 1, are homogeneous of degree zero. Moreover, Marshallian demands are homogeneous of degree zero in p and y

$$f_i(\mathbf{p}, y) = f_i(\theta\mathbf{p}, \theta y)$$

- Symmetry: this holds for Hicksian demands only, and states that

$$\frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = \frac{\partial h_j(\mathbf{p}, u)}{\partial p_i},$$

  which is a characteristic related to transitivity, and can be proved as follows:

$$h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i} \Rightarrow \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_i \partial p_j} = \frac{\partial^2 e(\mathbf{p}, u)}{\partial p_j \partial p_i} = \frac{\partial h_j(\mathbf{p}, u)}{\partial p_i}$$

- Negativity: the substitution matrix is negative semidefinite. It is so because this matrix is made of partial derivatives of Hicksian demands w.r.t. prices, and from symmetry, we know that this derivatives are actually second-order derivatives of the expenditure function, which is concave[12]. We have that

$$k' \cdot \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_n}{\partial p_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial p_n} & \cdots & \frac{\partial h_n}{\partial p_n} \end{bmatrix} \cdot k = \sum_{i=1}^{n} \sum_{j=1}^{n} k_i \cdot k_j \cdot \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} \leq 0$$

[12]The Hessian matrix of a concave function (like the cost function) is always symmetric and negative semidefinite.

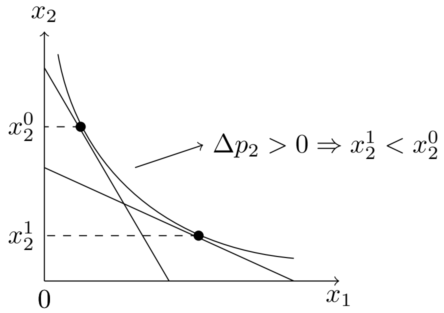
Moreover, we note that $\frac{\partial h_i}{\partial p_i} \leq 0,\ \forall i$, since they're the diagonal elements of a negative semidefinite matrix. Also, from homogeneity[13] (of degree zero in p) property, we have that

$$\sum_{j=1}^{n} p_j \cdot \frac{\partial h_i}{\partial p_j} = 0,\ \forall i.$$

6 Class 06

6.1 Law of Demand and the Slutsky Equation

The Law of Demand can be stated in terms of Hicksian demands as: "Hicksian (compensated) demands cannot be positively sloped". JR state it as: "A decrease in the own price of a normal good will cause quantity demanded to increase. If an own price decrease causes a decrease in quantity demanded, the good must be inferior".



Both statements reflect the idea that $\frac{\Delta x_i}{\Delta p_i} \leq 0$ always! Note that the law doesn't depend on convexity of the indifference curve, neither on its continuity.

From this, we can analyze the Slutsky Equation. Begin with one of the theorems derived from Duality, $h_i(\mathbf{p}, u) = f_i(\mathbf{p}, e(\mathbf{p}, u))$, where $f_i(\cdot)$ denotes the Marshallian demand. Let x_j^* be the Marshallian demand for good j evaluated at the optimal point. Then, the effect of a $\Delta p_j > 0$ over the Marshallian demand for good i is

$$\begin{aligned}
& h_i(\mathbf{p}, u) = f_i(\mathbf{p}, e(\mathbf{p}, u)) \overset{d/dp_j}{\Leftrightarrow} \\
& \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = \frac{\partial f_i(\mathbf{p}, y)}{\partial p_j} + \frac{\partial f_i}{\partial y} \cdot \frac{\partial e(\mathbf{p}, u)}{\partial p_j} \Leftrightarrow \\
& \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = \frac{\partial f_i(\mathbf{p}, y)}{\partial p_j} + \frac{\partial f_i}{\partial y} \cdot h_j(\mathbf{p}, u^*) \Leftrightarrow \\
& \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = \frac{\partial f_i(\mathbf{p}, y)}{\partial p_j} + \frac{\partial f_i}{\partial y} \cdot h_j(\mathbf{p}, v(\mathbf{p}, y)) \Leftrightarrow \\
& \frac{\partial h_i(\mathbf{p}, u)}{\partial p_j} = \frac{\partial f_i(\mathbf{p}, y)}{\partial p_j} + \frac{\partial f_i}{\partial y} \cdot x_j^* \Leftrightarrow \\
& \underbrace{\frac{\partial f_i(\mathbf{p}, y)}{\partial p_j}}_{\text{Total Effect}} = \underbrace{\frac{\partial h_i(\mathbf{p}, u)}{\partial p_j}}_{\text{Substitution Effect}} - \underbrace{x_j^* \cdot \frac{\partial f_i}{\partial y}}_{\text{Income Effect}}.
\end{aligned}$$

[13]Euler's Theorem (JR, p. 564): $f(\mathbf{x})$ is homogeneous of degree k if and only if

$$\sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} \cdot x_i = k \cdot f(\mathbf{x}).$$

In particular, for p_i, we have that

$$\frac{\partial f_i(\mathbf{p},y)}{\partial p_i} = \underbrace{\frac{\partial h_i(\mathbf{p},u)}{\partial p_i}}_{\text{¡ 0, diag element}} - x_i^* \cdot \underbrace{\frac{\partial f_i}{\partial y}}_{\text{- or +}}$$

If $\frac{\partial f_i(\mathbf{p},y)}{\partial y} < 0$, then the good in case is a Giffen good, i.e., demand for this good decreases as income increases. Mathematically, the income effect being negative implies that the total effect is positive, that is, demand for this good increases as its price increases, $\frac{\partial f_i(\mathbf{p},y)}{\partial p_i} > 0$.

Important: notice that, once $\frac{\partial f_i(\mathbf{p},y)}{\partial y} < 0$, every Giffen good is an inferior good. The converse is not true: it will depend on x_i^* being large enough so that $x_i^* \cdot \frac{\partial f_i}{\partial y} > \frac{\partial h_i(\mathbf{p},u)}{\partial p_i}$, and then the total effect will be positive. Economically, means that the demand for that good is so high that an increase in price forces the individual to buy more of that good, instead of substituting it for another one.

From this discussion, we provide an alternative Law of Demand: "For normal goods ($\frac{\partial f_i(\mathbf{p},y)}{\partial y} > 0$), the demand curve cannot be positively sloped". Income and substitution effects work together (same sign/direction); for inferior goods, those effects oppose each other. Thinking about income and price responses in the Slutsky Equation, we cannot actually see them, but we can estimate them empirically.

Further, we can define complementarity and substitutability (symmetric definitions) based on Hicksian demands:

$$\text{Complements: } \frac{\partial h_i(\mathbf{p},u)}{\partial p_j} < 0$$

$$\text{Substitutes: } \frac{\partial h_i(\mathbf{p},u)}{\partial p_j} > 0$$

Notice that it is not the same case with Marshallian demands because of the income effect, i.e., changes in prices affect Marshallian demands also indirectly, through income. And it may be the case that $\Delta p_2 > 0 \Rightarrow \Delta f_i < 0$.

We can also define the compensated price-elasticities with Hicksian demands:

$$\varepsilon_{ij}^* = \frac{\partial h_i(\mathbf{p},u)}{\partial p_j} \cdot \frac{p_j}{h_i(\mathbf{p},u)}.$$

From homogeneity (of degree zero in $\mathbf{p}$) of $h(\mathbf{p},u)$, we have that

$$\sum_j \frac{\partial h_i(\mathbf{p},u)}{\partial p_j} = 0 \Rightarrow \sum_j \varepsilon_{ij}^* = 0.$$

6.2 Hicksian Separability (DM, cap. 5, p. 119)

Assume that $(p_{k+1},\dots,p_n) = \pi \cdot (\bar{p}_{k+1},\dots,\bar{p}_n)$, where $(\bar{p}_{k+1},\dots,\bar{p}_n)$ are prices in a reference period. Then, definig $z = \sum_{i=k+1}^n \bar{p}_i x_i$ (= share of y dedicated to goods $x_{k+1},\dots,x_n$), the budget constraint can be written as

$$\sum_{i=1}^n p_i x_i = \sum_{i=1}^k p_i x_i + \pi \cdot \sum_{i=k+1}^n \bar{p}_i x_i = \sum_{i=1}^k p_i x_i + \pi \cdot z.$$

This means that prices in $(p_{k+1},\dots,p_n)$ are not relevant individually, i.e., since they move parallelly, we can look just to π in order to work with these prices as they (jointly) change. That's why we're able to replace them only by $\pi \cdot z$ in the budget constraint, considering all of them as only one thing.

Now, define $U_H(x_1,\dots,x_k,z)$ as

$$U_H(x_1,\dots,x_k,z) \equiv \max_{\{x_{k+1},\dots,x_n\}} \quad u(x_1,\dots,x_k,x_{k+1},\dots,x_n) \quad s.t. \quad z = \sum_{i=k+1}^n \bar{p}_i x_i.$$

This means that we can separate the maximization of goods $(x_{k+1}, \ldots, x_n)$ from the other goods, so that in U_H, the goods $(x_1, \ldots, x_k)$ are like parameters, and we are concerned only about the alocation of z on goods $(x_{k+1}, \ldots, x_n)$.

Further, if we maximize U_H, we get a new expression to the classical util. max. problem, such that both expressions are equivalent

$$\max \quad u \quad s.t. \quad \mathbf{p}'\mathbf{x} = y \Leftrightarrow$$

$$\max_{(x_1,\ldots,x_k,z)} \quad U_H(x_1, \ldots, x_k, z) \quad s.t. \quad y = \sum_{i=1}^{k} p_i x_i + \pi \cdot z \Leftrightarrow$$

$$\max_{(x_1,\ldots,x_k,z)} \quad \left\{ \max_{\{x_{k+1},\ldots,x_n\}} \quad u(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \quad s.t. \quad z = \sum_{i=k+1}^{n} \bar{p}_i x_i \right\} \quad s.t. \quad y = \sum_{i=1}^{k} p_i x_i + \pi \cdot z$$

In particular, in order to recover the optimum z^* from the cost function, we can derive $e(\cdot)$ only w.r.t. π, since π already contains all relevant information for prices in $(x_{k+1}, \ldots, x_n)$:

$$e^*(p_1, \ldots, p_k, \pi, u) = e(p_1, \ldots, p_k, \pi\bar{p}_{k+1}, \ldots, \pi\bar{p}_n, u)$$
$$\frac{\partial e^*(p_1, \ldots, p_k, \pi, u)}{\partial \pi} = \underbrace{\frac{\partial e}{\partial p_{k+1}}}_{x^*_{k+1}} \cdot \underbrace{\frac{\partial p_{k+1}}{\partial \pi}}_{\bar{p}_{k+1}} + \cdots + \underbrace{\frac{\partial e}{\partial p_n}}_{x^*_n} \cdot \underbrace{\frac{\partial p_n}{\partial \pi}}_{\bar{p}_n} = z^*.$$

7 Class 07

7.1 Hicksian Separability (cont'd)

One common application of Hicksian separability is Intertemporal Choice, where $\sum_{t=0}^{T} \beta^t \cdot u(c_t)$ is the utility function, and $u(c_t)$ is the instantaneous utility function. In this case, we're separating the multiple instantaneous util. functions over time, and agg them into one single function $f(U(c_1, \ldots, c_t)) = f(\sum_{t=0}^{T} \beta^t \cdot u(c_t))$.

Hicksian separability allows us to write intertemporal problems like separate points in time $(U(c_1, \ldots, c_t))$ in a more useful way. It comes from the budget constraint.

7.2 Weak Separability (DM, cap. 5, p. 120)

If it's possible to write the util. function $u(x_1, \ldots, x_n)$ as $U[v(x_1, \ldots, x_k), x_{k+1}, \ldots, x_n]$, where $v(\cdot)$ is called a "subutility function", we say that the preference is weakly separable in $(x_1, \ldots, x_k)$. If $(x_1^*, \ldots, x_n^*)$ maximizes u under $(p_1, \ldots, p_n)$, then:

$$(x_1^*, \ldots, x_k^*) = \max_{(x_1,\ldots,x_k)} v(x_1, \ldots, x_k) \qquad s.t. \qquad \sum_{i=1}^{k} p_i x_i = d^* = \sum_{i=1}^{k} p_i x_i^*,$$

where d^* is a function of k prices and income, $d^*(p_1, \ldots, p_n, y)$. In fact, $d^* < y$, since it's only the expenditure on the k goods at matter. Notice that the max problem is not constrained on $(x_{k+1}, \ldots, x_n)$, which means that utility is maximized regardlessly of the consumption of those goods. Thus, we can express the MRS_{ij}, with $i, j \leq k$ as

$$MRS_{ij} = \frac{(\partial U/\partial v) \cdot (\partial v/\partial x_i)}{(\partial U/\partial v) \cdot (\partial v/\partial x_j)} = \frac{\partial v/\partial x_i}{\partial v/\partial x_j},$$

i.e., MRS_{ij} doesn't depend on goods "outside" v. Thus, we have the following scenario with weak separability:

$$\begin{aligned}&\text{In general: } x_i^* = f(p_1, \ldots, p_n, y)\\&\text{With sep.: } x_i^* = f(p_1, \ldots, p_k, d^*(p_1, \ldots, p_n, y)), \forall i \leq k.\end{aligned}$$

Now, define $c(p_1, \ldots, p_k, v)$ as the cost function to the "sub" problem of maximizing v. Hence, we have the so called **two-stage budgeting**, which can be written as:

$$\max_{(v, x_{k+1}, \ldots, x_n)} U(v, x_{k+1}, \ldots, x_n) \qquad s.t. \qquad \underbrace{\overbrace{c(p_1, \ldots, p_k, v)}^{\text{first}} + \sum_{i=k+1}^{n} p_i x_i = y}_{\text{second}}\,.$$

From DM (p. 123), "[...] at the first or higher stage, expenditure is allocated to broad groups of goods, while at the second, or lower stage, group expenditures are allocated to the individual commodities. At each of these stages, information appropriate to that stage only is required. At the first stage, allocation must be possible given knowledge of total expenditure and appropriately defined group prices, while at the second stage, individual expenditures must be functions of group expenditure and prices within the group only. Both of these allocations have to be perfect in the sense that the result of two-stage budgeting must be identical to what would occur if the allocation were made in one step with complete information". Further, under some conditions (homotheticity), we also have that:

$$c(p_1, \ldots, p_n, v) = \phi(p_1, \ldots, p_n) \cdot v.$$

7.3 Strong or Additive Separability (DM, cap. 5, p. 137)

If it's possible to write the util. function $u(x_1, \ldots, x_n)$ as $v(x_1, \ldots, x_k) + g(x_{k+1}, \ldots, x_n)$, we say that the preference is strongly separable, or additive, in $(x_1, \ldots, x_n)$. Note carefully that it is preferences that are strongly or additively separable, not the utility function, hence

$$u = \prod exp[v_k(q_k)] \qquad \text{and} \qquad u = \sum v_k(q_k)$$

are both representations of the same additively separable preferences.

Moreover, note that since the function is additive, we can arbitrarily create new groups by combining any others, and this effectively prevents the existence of any particular relationships between pairs of groups.

7.4 Homothetic Preferences (DM, cap. 5, p. 142)

Under homotheticity, the MRS is constant for constant ratios of the goods, i.e., it depends only on the ratio of the demands. If relative prices are the same, the additional amounts of goods will keep proportional. More precisely, for any two bundles of goods $\{\mathbf{x}, \mathbf{y}\}$,

$$\begin{aligned}&\text{In terms of prefs.:} && \mathbf{x} \sim \mathbf{y} \Rightarrow \alpha\mathbf{x} \sim \alpha\mathbf{y}, \quad \forall \alpha \geq 0\\&\text{In terms of util.:} && u(\mathbf{x}) = u(\mathbf{y}) \Rightarrow u(\alpha\mathbf{x}) = u(\alpha\mathbf{y}), \quad \forall \alpha \geq 0\end{aligned}$$

Hence, for any ray from origin, all points in different indiff curves crossing this ray have the same MRS. The power of this definition is that if we know only one indiff curve generated by a homothetic pref, we're able to describe all indiff curves associated with this pref, because all indiff curves are increased/decreased versions of each other. Thus, we can fully describe the pref system that generated that indiff curve.

This property implies that the Engel curve (that shows the relation between the demand for one good and all levels of income) associated to a homothetic pref will be linear. Moreover, the income expansion path (that shows the chosen bundles of goods for different levels of income) will also be linear for homothetic prefs[14].

Homothetic preferences always admit a representation that is homogeneous of degree 1: $u(\mathbf{x}) = F[v(\mathbf{x})]$, where $v(\cdot)$ is homogeneous of degree 1 and $F[\cdot]$ is a monotonic transformation.

$$MRS_{ij} = \frac{\partial u/\partial x_i}{\partial u/\partial x_j} = \frac{\cancel{(\partial F/\partial v)}(\partial v/\partial x_i)}{\cancel{(\partial F/\partial v)}(\partial v/\partial x_j)} = \frac{\frac{\partial v(x_1,\ldots,x_n)}{\partial x_i}}{\frac{\partial v(x_1,\ldots,x_n)}{\partial x_j}} \overset{(\Delta)}{=} \frac{\frac{\partial v(x_1/x_k,\ldots,x_n/x_k)}{\partial x_i}}{\frac{\partial v(x_1/x_k,\ldots,x_n/x_k)}{\partial x_j}},$$

where in (Δ) we multiply the expression by $\frac{1/x_k}{1/x_k}$ and use the fact that $\frac{\partial v}{\partial x_i}$ is homogeneous of degree zero, since v is homogeneous of degree 1, and thus the last equality holds.

Further, under homotheticity, all income-elasticities are equal to 1, $\eta_i = 1, \forall i$. Since the MRS is always the same across the indiff curves, if income increases, demand for both (or whatever amount of) goods also increases, but in the same proportion of $\Delta y > 0$ for all goods; otherwise, the MRS would not be equal through all the indiff map. This means that keeping the MRS constant through indiff curves implies keeping this rate constant through all levels of income too.

With homothetic preferences, we can always choose $v(\cdot)$ as the representation of preferences:

$$c(p_1,\ldots,p_k,v) = \min\left\{\sum_{i=1}^{k} p_i x_i \quad s.t. \quad v(\mathbf{x}) \geq v\right\} = \phi(p_1,\ldots,p_k)\cdot v.$$

This representation is useful because provides an expression that is homogeneous of degree 1. Thus, with homothetic preferences, we can write

$$\max_{(v,x_{k+1},\ldots,x_n)} U(v,x_{k+1},\ldots,x_n) \qquad s.t. \qquad \phi(p_1,\ldots,p_k)\cdot v + \sum_{i=k+1}^{n} p_i x_i = y.$$

Finally, homothetic preferences are related to separability in the following way. First, because the income expansion path is a straight line, it follows that the composition of the budget is independent of total expenditure or of utility; it depends only on the ratio of demands between the goods in case. Second, w.r.t. the structure of the cost function, if we label indiff curves along any ray through the origin such that double utility is generated by being twice as far from the origin, then the cost of reaching utility u must be proportional to u, i.e., twice higher.

8 Class 08

8.1 Aggregation (DM, cap. 6)

Aggregation is a first approach to deal with multiple agents in microeconomics. Initially, we can write

$$X(\mathbf{p}, y^1,\ldots,y^N) \equiv \sum_{i=1}^{N} x^i(\mathbf{p}, y^i),$$

where superscripts refer to consumers; $x^i(\mathbf{p}, y^i)$ is the Marshallian demand of the i-th consumer; N is the number of individuals in the society; and X is a $(n \times 1)$ vector.

When is it that we can write $X(\mathbf{p}, y^1,\ldots,y^N) = F(\mathbf{p}, Y)$, with $Y = \sum_{i=1}^{N} y^i$? Intuitively, by doing this, we lose the information of income distribution within the society, and we only deal with aggregate

[14]Notice that all classical preference types are homothetic, except the quasilinear preferece.

income. If we're able to this moviment, it must be the case that income is not relevant for our study, or that Δy doesn't change demand. In this case, we need that income distribution to be irrelevant for demand, in order to this to work. In general, this requires homotheticity, because income-elasticity equals 1 for all goods, i.e., $\eta_j = 1, \forall j$. Also, all agents must have the same homothetic preferences.

Under some restrictions, a slightly more flexible form also works:

$$x_j^i = \alpha_j^i(\mathbf{p}) + \beta_j(\mathbf{p}) \cdot y^i,$$

where i represents the i-th consumer, and j is the j-th good. This is called quasi-homotheticity, or Gorman form. Notice that homothetic preferences have a shape like $x_j^i = \beta_j(\mathbf{p}) \cdot y^i$, where income-elasticity $\eta_i = \beta_j(\mathbf{p}) \cdot \frac{y^i}{x_j^i} = \beta_j(\mathbf{p}) \cdot \frac{y^i}{\beta_j(\mathbf{p} \cdot y^i)} = 1$.

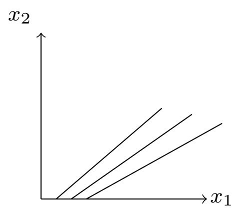
In the quasi-homothetic case, η_i is not exactly 1, because there's that α allowing some heterogeneity among consumers (rich and poor ones). DM refers to that α_j^i as a subsistence level of expenditure in good j by the agent i, i.e., a idiosyncratic fixed cost. Thus, for quasi-homothetic preferences, η_i is

$$\eta_i = \beta_j(\mathbf{p}) \cdot \frac{y^i}{\alpha_j^i(\mathbf{p}) + \beta_j(\mathbf{p}) \cdot y^i},$$

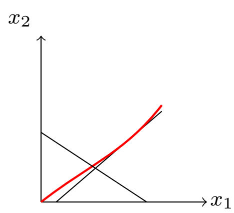
and this form tends to 1, and thus a homothetic preference, as y^i grows[15]. It means that, for a sufficiently high level of y, the agent i will be "free" from considering his subsistence level of expenditure in good j.

This Gorman form still works, because β_j doesn't depend on each agent (note that there's no i in β_j). Thus, changing y among consumers (taking from one and giving to another one), in the margin, doesn't change nothing in aggregate terms.

Moreover, under homotheticity, the income-expansion path (Engel curve) must be linear always, while non-homothetic preferences don't have this requirement. By including $\alpha_j^i(\mathbf{p})$, we allow for income-expansion paths (Engel curves) that do not pass over the origin.



But note that in the Gorman form, even if $y^i = 0$, the guy i still can have positive demand by $\alpha_j^i(\mathbf{p})$ that doesn't depend on y^i, which makes no sense. Thus, the Gorman form cannot be strictly true all the time. It's an indirect way to model preferences that are not actually like this, but at some point is a nice approximation.



[15] DM (p. 144-145) says that, under homothetic preferences, Engel curves (expenditure path as income grows) are straight lines through origin, so $\eta_i = 1$ for all i; under quasi-homotheticity, the straight lines need not to go through the origin, and thus η_i only **tend** to 1 as total expenditure (or income) increases.

The red line is the real income-expansion path, and the estimate one is in black. It's hard to model that first part close to the origin, but above the BC, the estimate is good.

This works with some on the support $(y^i, \mathbf{p})$. Also, Gorman form will look/behave like a demand function with these restrictions.

8.2 Integrability (JR, DM)

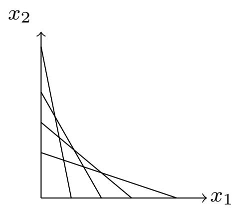
The integrability problem is to recover a consumer's utility function from his demand function, i.e., given a vector-valued function of prices and income (demand function), under which conditions we may say that there is a utility function that generated it as its demand function. This leads to the integrability theorem.

Theorem (Integrability): a continuous and differentiable function $x : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n_+$ is a demand function generated by some utility function that is strictly monotone, strictly quasi-concave, and continuous iff it satisfies addivity, symmetry, and negative semi-definitness of the Slutsky Matrix (homogeneity of degree zero is implied by these 3 properties).

The proof of this theorem can be found in JR, p. 88-90, and is somewhat intuitive. The intuition over the function x is that the $(n+1)$ elements are n prices plus 1 income. But in order to grasp it, it is useful to understand another theorem (from Duality).

Theorem. Consider some function $E(\mathbf{p}, u)$, continuous, strictly increasing, homogeneous of degree 1, and concave on $\mathbf{p}$. Then, $E(\mathbf{p}, u)$ is the cost function some utility function.

The idea is that $E(\mathbf{p}, u)$ is actually a function that "draws" a lot of BC lines, focusing on lowering costs in order to achieve a indifference curve. In the end, we come up with a indifference curve close to something well-behaved. This is kind of what the proof of this theorem does.



Imagine an indifference curve going over all these BC lines. This theorem creates this approximation to any weird indifference curve (not necessarily equivalent to this approximation) that touches some of these BC lines, even if it touches just one of them. By duality, a demand function (that vector-valued function) satisfying the conditions on the first theorem will have an associated utility function that generated it.

9 Class 09

9.1 Revealed Preferences (JR)

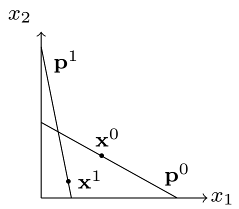
From JR, the basic idea is simple: if the consumer buys one bundle instead of another affordable bundle, then the first bundle is considered to be revealed preferred to the second. Instead of axioms, with this information we make assumptions about the consistency of the choices that are made. Formally, we have the following.

Weak Axiom of Revealed Preferences (WARP): choices satisfy the WARP if, for all pair of bundles $\{\mathbf{x}^0, \mathbf{x}^1\}$, where $\mathbf{x}^0$ is chosen under $\mathbf{p}^0$, and $\mathbf{x}^1$ is chosen under $\mathbf{p}^1$,

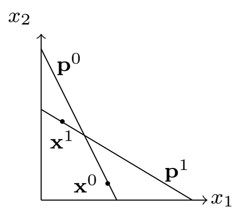
$$\mathbf{p}^0 \cdot \mathbf{x}^1 \leq \mathbf{p}^0 \cdot \mathbf{x}^0 \Rightarrow \mathbf{p}^1 \cdot \mathbf{x}^0 > \mathbf{p}^1 \cdot \mathbf{x}^1.$$

In this case, we write $\mathbf{x}^0 R^D \mathbf{x}^1$, and say that $\mathbf{x}^0$ is "directly revealed preferred" to x^1. The WARP says that, if under the initial prices $\mathbf{p}^0$, the consumer chose $\mathbf{x}^0$ instead of $\mathbf{x}^1$, and $\mathbf{x}^1$ was affordable, then it must be the case that, under different prices $\mathbf{p}^1$, this consumer considers $\mathbf{x}^0$ as better than $\mathbf{x}^1$ also. Moreover, if $\mathbf{x}^1$ is chosen under $\mathbf{p}^1$, then it must be the case that, under $\mathbf{p}^1$, $\mathbf{x}^0$ is not affordable.

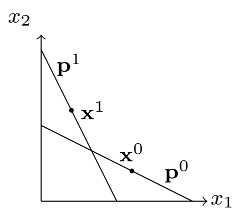
The following graph represents consistent choices, in terms of WARP. The individual chose $\mathbf{x}^0$ firstly, when prices were $\mathbf{p}^0$; in that scenario, $\mathbf{x}^1$ affordable, but was not chosen. Later on, when prices became $\mathbf{p}^1$, $\mathbf{x}^0$ wasn't affordable anymore, although $\mathbf{x}^0 R^D \mathbf{x}^1$; thus, he chooses $\mathbf{x}^1$.



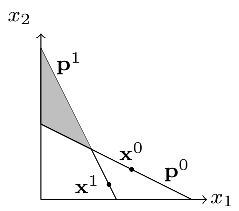
This next graph shows an inconsistent system of choice, in terms of WARP. The individual chose $\mathbf{x}^0$ when prices were $\mathbf{p}^0$. In that scenario, $\mathbf{x}^1$ was affordable, and thus, $\mathbf{x}^0 R^D \mathbf{x}^1$. But when prices became $\mathbf{p}^1$, he chose $\mathbf{x}^1$, even though $\mathbf{x}^0$ was still affordable and directly revealed preferred to $\mathbf{x}^1$.



Finally, note that there's nothing to say about this following paragraph. Here, the agent chose $\mathbf{x}^0$ when $\mathbf{x}^1$ was not affordable; on the other hand, he chose $\mathbf{x}^1$ when $\mathbf{x}^0$ was not affordable either. There's nothing to say because we cannot say that one bundle is directly revealed preferred to another.



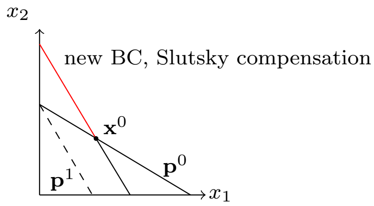
Strong Axiom of Revealed Preferences (SARP): choices satisfy SARP if, for all sequences of bundles $\{\mathbf{x}^0, \dots, \mathbf{x}^k\}$, where $\mathbf{x}^0 R^D \mathbf{x}^1$, $\mathbf{x}^1 R^D \mathbf{x}^2$, …, $\mathbf{x}^{k-1} R^D \mathbf{x}^k$, then $\mathbf{x}^k$ is not $R^D \mathbf{x}^0$. Moreover, $\mathbf{x}^0 R \mathbf{x}^k$, and we say that $\mathbf{x}^0$ is "revealed preferred" to $\mathbf{x}^k$.



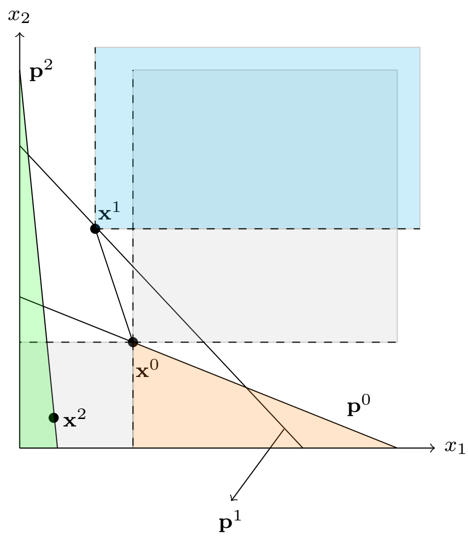
SARP says that $\mathbf{x}^0$ is revealed preferred (by $\mathbf{x}^1$) to any bundle in the gray area, because $\mathbf{x}^0 \succ \mathbf{x}^1$, and $\mathbf{x}^0 \succ \mathbf{x}^*$, where $\mathbf{x}^*$ is any bundle in the gray area. Note that we can say that even though $\mathbf{x}^*$ and $\mathbf{x}^0$ are not related at all.

If choices satisfy WARP and are on the BC, they are consistent with the negativity of the Slutsky matrix, and with homogeneity. If, additionally, they satisfy SARP, they are consistent with preferences that are complete and transitive (thus, attend symmetry too).

Slutsky Compensation. Is a way of providing the agent with the same bundle chosen before a $\Delta\mathbf{p} > 0$, by adjusting his BC. It converges to the Hicksian compensation when $\Delta\mathbf{p}$ is small. Recall: Hicksian compensation only focuses on bringing the individual back to the same utility level (i.e., the same indiff curve), not necessarily to the same bundle on that util. level. The next graph illustrates the Slutsky compensation for a $\Delta p_1 > 0$. Note that if choices are consistent with WARP, then the choice must be on red part of the new BC, because on that part we have all bundles that were not affordable under $\mathbf{p}^0$ but are still better than $\mathbf{x}^0$ (under $\mathbf{p}^0$), and now these bundles are affordable under the new BC (and satisfy additivity, since are on the frontier of the BC), as well as $\mathbf{x}^0$.



Recoverability. This is approach to the problem of recover the shape of indifference curves via BC reflecting different price levels. All we have in this situation is the information about the choices the individual makes under different prices. In recoverability, we assume that preferences are strictly monotone and convex.



Start with the information that, under $\mathbf{p}^0$, the agent chose $\mathbf{x}^0$. We already know that the indifference curve won't pass through any of the gray areas, because under $\mathbf{x}^0$ he is worse, and above $\mathbf{x}^0$ bundles are not affordable.

With the information that, under $\mathbf{p}^1$, the agent chose $\mathbf{x}^1$, we know that indiff curve will not pass through any point between $\mathbf{x}^0$ and $\mathbf{x}^1$, since preferences are convex, and thus any convex ("weighted average") combination of $\mathbf{x}^0$ and $\mathbf{x}^1$ is better than $\mathbf{x}^0$. Also, by $\mathbf{x}^1$, we know that the indiff curve will not pass through the blue area, because all bundles there are not affordable under $\mathbf{p}^1$.

With the information that, under $\mathbf{p}^2$, the agent chose $\mathbf{x}^2$, we rule out the green triangle from a possible area where the indiff curve would pass, since there the agent is worse not only w.r.t. $\mathbf{x}^2$, but also w.r.t. the previous choices (note that this implies preferences that complies with the WARP, under this drawing).

Note that we can also rule out the orange triangle under $\mathbf{x}^0$, since there the agent is also worse than $\mathbf{x}^0$, under $\mathbf{p}^0$. And so on, until we have enough information over choices under different set of prices to eliminate as much areas as possible, making the indiff curve arise.

10 Class 10

10.1 Neoclassical Firm Theory, basics (JR, cap. 3)

Production Function. It a function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, continuous, strictly increasing, and strictly quasi-concave in $\mathbb{R}^n_+$, where $f(0) = 0$. Also,

$$\text{Vector of inputs: } \equiv \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \text{Output: } \equiv y.$$

Sometimes, instead of a production function, we see a definition of a production possibility set (like in MWG): $Y \subseteq \mathbf{R}^m$; $Y = (y_1, \dots, y_m)$, with positive and negative entries.

Isoquant: $Q(y) \equiv \{\mathbf{x} \geq 0 \mid f(\mathbf{x}) = y\}$.

$\boldsymbol{MRTS_{ij}}$: Marginal rate of technical substitution,

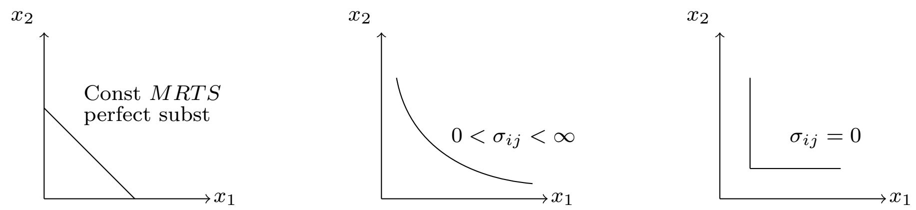
$$MRTS_{ij} = \frac{\partial f(\mathbf{x})/\partial x_i}{\partial f(\mathbf{x})/\partial x_j},$$

where $\frac{\partial f(\mathbf{x})}{\partial x_i}$ is the marginal productivity of x_i.

Elasticity of substitution between $\mathbf{x}_i$ and $\mathbf{x}_j$: let $f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i}$. Then,

$$\sigma_{ij} = \frac{d\,ln(x_i/x_j)}{d\,ln(f_i(\mathbf{x})/f_j(\mathbf{x}))} = \frac{d(x_i/x_j)}{(x_i/x_j)} \cdot \frac{f_i(\mathbf{x})/f_j(\mathbf{x})}{d(f_i(\mathbf{x})/f_j(\mathbf{x}))} = \frac{[f_i(\mathbf{x})/f_j(\mathbf{x})] \;/\; (x_i/x_j)}{d[f_i(\mathbf{x})/f_j(\mathbf{x})] \;/\; d(x_i/x_j)},$$

where $\frac{1}{d(f_i(\mathbf{x})/f_j(\mathbf{x}))}$ on the third step is the marginal productivity (?).



In the first graph, since x_1 and x_2 are perfect substitutes, then the numerator of the last expression goes to ∞ ($MRTS = \frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} \to \infty$), and thus $\sigma_{ij} \to \infty$.

In the third graph, since x_1 and x_2 are perfect complements, then the numerator of the last expression is zero, and thus $\sigma_{ij} = 0$.

In the second graph, note that x_1 and x_2 have a "normal" relation in terms of substitutability, which imples that $0 < \sigma_{ij} < \infty$.

Consider the following CES production function $y = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$. In this case, we have that

$$\sigma = \frac{1}{1-\rho} \Rightarrow \begin{cases} \rho = 0 \Rightarrow \sigma = 1 \longrightarrow \text{Cobb-Douglas} \\ \rho = 1 \Rightarrow \sigma \to \infty \longrightarrow \text{Perfect Substitutes} \\ \rho \to -\infty \Rightarrow \sigma = 0 \longrightarrow \text{Leontieff} \end{cases}$$

Returns to Scale. These are global concepts that apply when we vary all inputs at the same time. We have the following possibilities:

(i) Constant returns to scale: $f(t\mathbf{x}) = t{\cdot}f(\mathbf{x})$, $\forall t > 0$, $\forall \mathbf{x}$. Thus, CRS prod. functions are homogeneous of degree 1;

(ii) Increasing returns to scale: $f(t\mathbf{x}) > t \cdot f(\mathbf{x})$, $\forall t > 1$, $\forall \mathbf{x}$. Thus, IRS prod. functions are homogeneous of degree ¿ 1;

(iii) Decreasing returns to scale: $f(t\mathbf{x}) < t \cdot f(\mathbf{x})$, $\forall t > 1$, $\forall \mathbf{x}$. Thus, DRS prod. functions are homogeneous of degree ¡ 1.

Notice that it wouldn't make sense define similar concepts for util. functions, since util. functions and util. levels have no meaning at all. Prod. functins do have a meaning: the amount of output a firm will produce given some level of inputs.

Again, these are global concepts. Locally, we can define similar concepts base on the following relation

$$\mu(\mathbf{x}) = \lim_{t\to 1} \frac{d\ ln[f(t\mathbf{x})]}{d\ ln(t)} = \frac{\sum\limits_{i=1}^{n} f_i(\mathbf{x}) \cdot x_i}{f(\mathbf{x})} \lesseqgtr 1.$$

If the prod function is homogeneous of degree 1, the above expression equals 1, i.e., $\sum_{i=1}^{n} f_i(\mathbf{x}){\cdot}x_i = f(\mathbf{x})$.

10.2 Firm's Cost Minimization (JR, section 3.3, p. 136)

Assuming that prices are given, and that the firm is competing for inputs in the market, we have that its cost minimization problem is

$$c(\mathbf{w}, y) \equiv \min_{\mathbf{x}\in\mathbb{R}^n_+} \{\mathbf{w}'\mathbf{x} \qquad s.t. \qquad f(\mathbf{x}) = y\},$$

where $c(\mathbf{w}, y)$ is the cost function, and $\mathbf{w}' = {}_{w_1,\ldots,w_n}$ gives the price of inputs. The only assumption is that the firm behaves competitively in the market of inputs, i.e., $\precapprox$ is given.

The solution to this problem is $\mathbf{x} = \mathbf{x}(\mathbf{w}, y)$, which is the conditional demand for $\mathbf{x}$. Another notation is $c(\mathbf{w}, y = \mathbf{w}' \cdot \mathbf{x}(\mathbf{w}, y)$.

Let $\mathbf{x}^*$ be the solution to this problem. The FOCs are:

$$\begin{aligned} w_i &= \lambda \cdot \frac{\partial f(\mathbf{x}^*)}{\partial x_i}, \qquad i = 1, \ldots, n \\ f(\mathbf{x}) &= y. \end{aligned}$$

Because $w_i > 0, i = 1, \ldots, n$, we may divide the preceding i-th equation by the j-th to obtain the $MRTS_{ij}$:

$$MRTS_{ij} = \frac{\partial f(\mathbf{x}^*)/\partial x_i}{\partial f(\mathbf{x}^*)/\partial x_j} = \frac{w_i}{w_j}.$$

Further, we have the following properties over the cost function:

1) $c(\mathbf{w}, 0) = 0$

2) Strictly increasing on y, i.e., more production leads to more expenditure;

3) Increasing on $\mathbf{w}$, i.e., costs don't decrease if $\Delta\mathbf{w} > 0$;

4) Homogeneous of degree 1 on $\mathbf{w}$, i.e., if input prices increase by $t > 0$, costs of production will increase by t;

5) Concave on $\mathbf{w}$, i.e., if $\Delta\mathbf{w} > 0$ and firm does nothing, then costs raise linearly (1 to 1) with $\mathbf{w}$ (property 4);

6) Shephard's lemma:

$$\frac{\partial c(\mathbf{w}, y)}{\partial w_i} = x_i(\mathbf{x}, y)$$

Also, w.r.t. the conditional demand for inputs, we have the following properties:

1) $\mathbf{x}(\mathbf{w}, y)$ is homogeneous of degree zero on $\mathbf{w}$;

2) The substitution matrix between inputs is negative semi-definite and symmetric.

$$\sigma^*(\mathbf{w}, y) \equiv \begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \cdots & \frac{\partial x_1}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial w_1} & \cdots & \frac{\partial x_n}{\partial w_n} \end{bmatrix}$$

In particular, this implies that the elements in the main diagonal are negative ($\frac{\partial x_i(\mathbf{w},y)}{\partial w_i} < 0, \forall i$), i.e., if the price of an input increases, the demand for it decreases.

11 Class 11

11.1 Homothetic Production Functions (JR, theorem 3.4, p. 140)

If a production function $F(\mathbf{x})$ is homothetic, then it is also multiplicatively separable in prices, such that

$$c(\mathbf{w}, y) = h(y) \cdot c(\mathbf{w}, 1)$$

$$\mathbf{x}(\mathbf{w}, y) = h(y) \cdot \mathbf{x}(\mathbf{w}, 1),$$

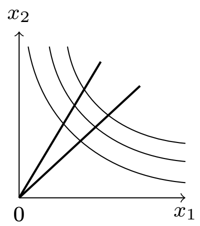
where $c(\mathbf{w}, 1)$ is the cost of 1 unit of output, $\mathbf{x}(\mathbf{w}, 1)$ is the conditional input demand for 1 unit of output, and $h(y)$ is strictly increasing.

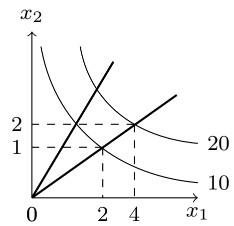
Moreover, if a production function is homogeneous of degree $\alpha > 0$, then

$$c(\mathbf{w}, y) = y^{1/\alpha} \cdot c(\mathbf{w}, 1)$$

$$\mathbf{x}(\mathbf{w}, y) = y^{1/\alpha} \cdot \mathbf{x}(\mathbf{w}, 1).$$

Notice that homogeneous production functions will also be homothetic, but the converse is not always true.





The first graph illustrates homotheticity for general functions (it could be utility functions), and we note that there's nothing about the level of each curve, thus they are homothetic but not necessarily homogeneous.

In the second graph, we are considering isoquants. Here, homogeneity tells the same of homotheticity (i.e., the slope of the level curves), plus the "level" of each curve. In this case, is a homogeneous of degree 1 production function, because we can check that in order to double the production level, all inputs had to be doubled. Hence,

- Homogeneous of degree 1 production functions have constant returns to scale, i.e., $c(\mathbf{w}, y) = y \cdot c(\mathbf{x}, 1)$;

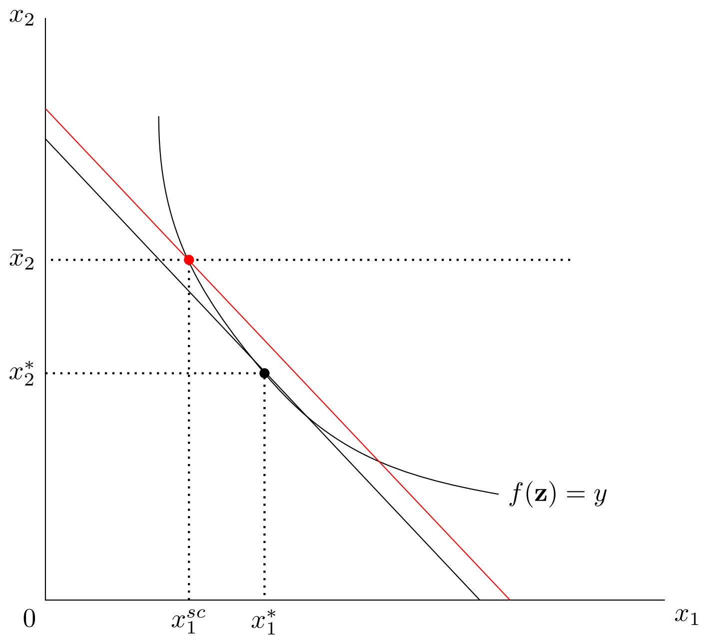
- Homogeneous of degree 2 (or greater) production functions have increasing returns to scale, i.e., $c(\mathbf{w}, y) = y^{1/2} \cdot c(\mathbf{x}, 1)$, meaning that it is possible to achieve the double of production without necessarily doubling the amount of all inputs.

11.2 Short Run and Long Run (JR, definition 3.6, p. 143)

Short run cost function. Let $z(\mathbf{x}, \bar{\mathbf{x}})$ be a vector of inputs, where $\mathbf{x}$ is a subvector of variable inputs, and $\bar{\mathbf{x}}$ is a fixed subvector of inputs. Also, let $\mathbf{w}$ and $\bar{\mathbf{w}}$ be the price vectors for $\mathbf{x}$ and $\bar{\mathbf{x}}$, respectively. The short-run, or restricted, total cost function function is defined as

$$sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) = \min_{\mathbf{x}} \{\mathbf{w}'\mathbf{x} + \bar{\mathbf{w}}'\bar{\mathbf{x}} \qquad s.t. \qquad f(\mathbf{z}) \geq y\}.$$

If $\mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}})$ solves the problem, then $sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) = \mathbf{w}'\mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}) + \bar{\mathbf{w}}'\bar{\mathbf{x}}$, where $\mathbf{w}'\mathbf{x}(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}})$ is the variable cost, and $\bar{\mathbf{w}}'\bar{\mathbf{x}}$ is the fixed cost.



The graph above shows the idea of fixed cost with the firm's cost minimization problem (CMP). The solution to the CMP is at point (x_1^*,x_2^*). This point is on the isocost curve $c(w_1, w_2, y)$, where there are no fixed costs and the firm is able to choose any amount of inputs. But in the short run, that are fixed costs; in this case, the level of x_2 is fixed in $\bar{x}_2$, which represents a fixed cost given by the expenditure to have $\bar{x}_2$. We notice that this level os $\bar{x}_2$ is fixed for every choice of x_1, i.e., given $\bar{x}_2$, the firm is free to choose only the level x_1. Thus, in order to produce the output level of y, the firm faces another isocost curve, $sc(w_1, \bar{w}_2, y; \bar{x}_2)$ (the red one). This new isocost curve provides a different set of costs than the first one. Note that since prices are the same for both isocost curves, the slope of them is the same. Hece, in order to produce $f(\mathbf{z}) = y$, the firm must choose the level x_1^{sc} of the input x_1, because it represents the lower cost, given $\bar{x}_2$, to achieve y.

One important insight that arises is that short-run costs can never be lower than long-run costs. For a given y, the $sc(\cdot)$ function can never be lower than the $c(\cdot)$ function, and they coincide only at the minimum of $sc(\cdot)$ (for a given y).

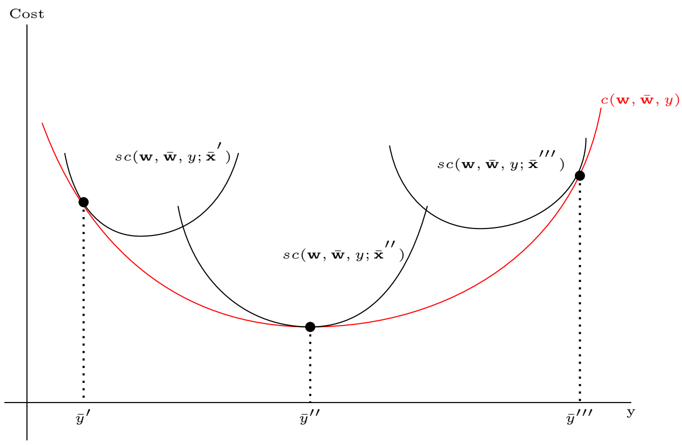
Now, define $\bar{\mathbf{x}}(y)$ as the level of $\bar{\mathbf{x}}$ that minimizes the short-term costs[16]. By definition,

$$\frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial \bar{x}_i} = 0,$$

since $\bar{\mathbf{x}}(y)$ is the minimum solution. We also know that: $[c(\mathbf{w}, \bar{\mathbf{w}}, y) = sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))]$[17]. Thus,

$$\frac{\partial c(\mathbf{w}, \bar{\mathbf{w}}, y)}{\partial y} = \frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial y} + \sum_i \underbrace{\frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y))}{\partial \bar{\mathbf{x}}_i}}_{=0} \cdot \frac{\partial \bar{\mathbf{x}}_i(y)}{\partial y} = \frac{\partial sc(\mathbf{w}, \bar{\mathbf{w}}, y; \bar{\mathbf{x}}(y)))}{\partial y}.$$

Note that, once $\bar{\mathbf{x}}(y)$ represents the minimum and fixed cost to achieve y, the partial derivative of $sc(\cdot)$ w.r.t. to $\bar{x}_i$ is zero, because $\bar{x}_i$ is fixed (constant) for all i. Hence, we have that the partial derivative of the long-run cost function w.r.t. y is equal to the partial derivative of the short-run cost function w.r.t. y evaluated at the minimum/optimal amounts $\bar{\mathbf{x}}(y)$. From this, using the envelope theorem, we can check that the long-run cost curve is the lower envelope of the short-run cost curve (JR, p. 143).



This graph is actually about average costs, but it is good to understand the idea. The long-run curve is the one in red. For every point of the long-run, there is a short-run curve that minimizes fixed costs associated with that point; in this case, we have three short-run curves, each one minimizing costs for given levels of output $(\bar{y}', \bar{y}'', \bar{y}''')$. Hence, there are infinite short-run cost curves that achieve each $\bar{y}$.

12 Class 12

12.1 Perfect Competition (JR, section 3.5, p. 145)

Perfect competition means only that firms takes prices as given.

We're now concerned with the profit maximization problem. It will be

$$\max_{\mathbf{x}, y} \{p \cdot y - \mathbf{w}'\mathbf{x} \qquad s.t. \qquad f(\mathbf{x} \geq y)\}.$$

Under monotonicity, the constraint is satisfied with equality, $f(\mathbf{x}) = y$, thus

$$\max_{\mathbf{x}} \{p \cdot f(\mathbf{x}) - \mathbf{w}'\mathbf{x}\}.$$

[16] In the previous case, $\bar{\mathbf{x}}_2(y) = x_2^*$, which is equal to long-term costs.

[17] This expression tells us, mathematically, the story that the red-line will never be below the black line, and that red-line=black-line only when the short-term cost (red-line) is evaluated at the minimum point of $\bar{\mathbf{x}}(y)$.

The FOCs are n equations (one for each input) of the form

$$p \cdot \frac{\partial f(\mathbf{x})}{\partial x_i} = w_i, \quad x_i^* > 0, \quad \forall i.$$

Further, the marginal productiviy of x_i is given by: $MgP_i = \frac{\partial f(\mathbf{x})}{\partial x_i}$. Dividing FOCs of the i-th and the j-th input yields in the $MRTS$:

$$MRTS_{ij} = \frac{\partial f(\mathbf{x})/\mathbf{x}_i}{\partial f(\mathbf{x})/\mathbf{x}_j} = \frac{w_i}{w_j}.$$

We can think about this problem in two steps:

1. Firm minimizes cost for every level of production y and every inputs price vector $\mathbf{w} \Rightarrow c(\mathbf{w}, y)$;
2. Firm chooses the level of production y to maximize profits.

Based on this, we can rewrite the problem as:

$$\max_y \{p \cdot y - c(\mathbf{w}, y)\}.$$

But since prices are given, i.e., $\mathbf{w}$ is fixed, we can simply write

$$\max_y \{p \cdot y - c(y)\}.$$

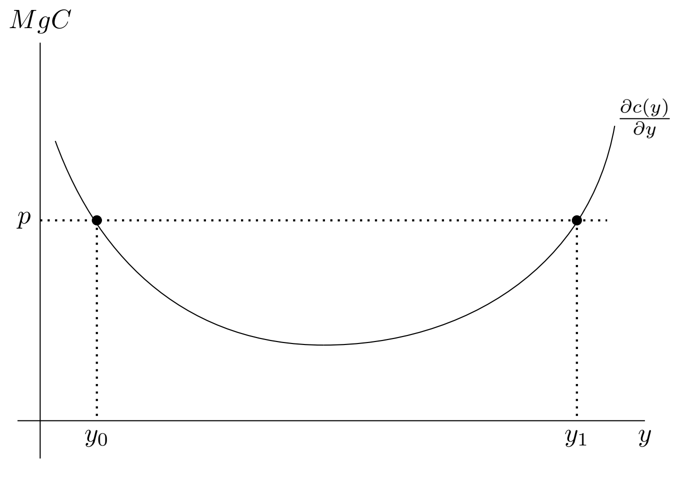
The FOC of this new problem is:

$$p - \frac{\partial c(y)}{\partial y} = 0 \Rightarrow p = \frac{\partial c(y)}{\partial y} = MgC(y) = \text{Marginal cost},$$

and the SOC is:

$$-\frac{\partial^2 c(y)}{\partial y^2} < 0 \Rightarrow \frac{\partial^2 c(y)}{\partial y^2} > 0$$

which means that the marginal cost of producing y must be increasing, in order to this solution to be a maximum.

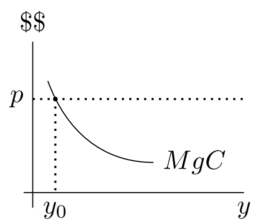


In this graph, p is the given market price for the good the firm produces; y_0 and y_1 are two levels of production that the firm may choose to sell it for the price p. But notice that, at y_0, although the FOC is satisfied, the SOC is not: marginal costs are decreasing at the point (p, y_0). On the other hand, at y_1, either FOC and SOC are satisfied: marginal costs are increasing at the point (p, y_1). Thus, SOC tells us what happens with marginal costs when producing a little bit more; to be a maximum, this cost must increase.

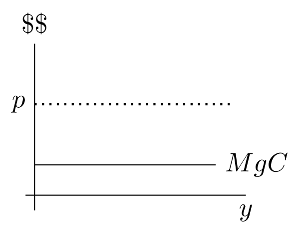
Profit funcion. It can be expressed as $\pi(p, \mathbf{w})$. If the solution to the profit maximization problem are $y(p, \mathbf{w})$ (supply function) and $\mathbf{x}(p, \mathbf{w})$ (demand function for inputs), then

$$\pi(p, \mathbf{w}) = p \cdot y(p, \mathbf{w}) - \mathbf{w}'\mathbf{x}(p, \mathbf{w}) \qquad \Leftrightarrow \qquad \pi(p, \mathbf{w}) = \max_{\{y, \mathbf{x}\}} \ \{p \cdot y(p, \mathbf{w}) - \mathbf{w}'\mathbf{x}(p, \mathbf{w}) \quad s.t. \quad f(\mathbf{x}) = y\}.$$

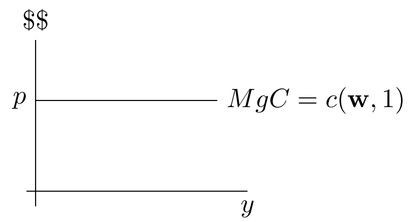
where $\mathbf{w}'$ is the inputs price vector transposed, just to allow that product.



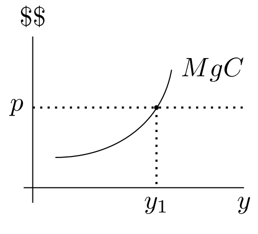
With increasing returns to scale (graph above), we note that it is not possible to have perfect competition, because y_0 doesn't attend the SOC. A firm in this situation will choose to grow indefinitely, and will eventually take all the market.



With constant returns to scale and marginal cost inferior to the market price (graph above), the firm will choose to increase production indefinitely. Eventually, something happens that makes returns to scale not constant anymore.



With constant returns to scale and marginal cost equal to the market price (graph above), we note that there will be free entry of new firms and zero profit. Weird, because the size of production is not defined, thus the firm is indifferent to producing or not. But this model is useful to analyze solutions where the size of the firm is not important.



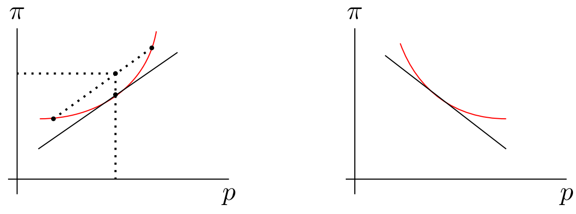
Finally, with decreasing returns to scale (graph above), we note that the profit function $\pi(p.\mathbf{w})$ is well defined, and has the following properties analogous to the properties of the indirect utility function (think about the envelope theorem):

1) Increasing in p: from the envelope theorem;

2) Decreasing in $\mathbf{w}$: from the envelope theorem;

3) Homogeneous of degree 1 in $(\mathbf{w}, p)$: $\pi(t\mathbf{w}, tp) = t \cdot \pi(\mathbf{w}, p), \forall t > 0$;

4) Convex in $(p, \mathbf{w})$: if prices increase, and firm is not incompetent (it has the red-line, left graph), it will increase its profits more than linearly. On the other hand, if costs of inputs increase, and firm is not incompetent (it has the red-line, right graph), it will avoid reduce its profits linearly.



5) Hotelling Lemma:

$$\frac{\partial \pi(p, \mathbf{w})}{\partial p} = y(p, \mathbf{w}) \quad \text{and} \quad \frac{\partial \pi(p, \mathbf{w})}{\partial w_i} = -x_i(p, \mathbf{w}), \ \forall i.$$

It says that π is decreasing in prices of inputs (from the envelope theorem), and not decreasing in the price of the good being produced[18].

Further, we have the following properties of the supply function and the demand for inputs:

1) Homogeneous of degree zero in $(p, \mathbf{w})$: this is because they're derivatives of the profit function, which is homogeneous of degree 1

$$y(tp, t\mathbf{w}) = y(p, \mathbf{w}), \ \forall t > 0$$

$$\mathbf{x}(tp, t\mathbf{w}) = \mathbf{x}(p, \mathbf{w}), \ \forall t > 0$$

2) The substitution matrix is symmetric and positive semi-definite, since it is the Hessian matrix of a convex function.

$$\begin{bmatrix} \frac{\partial y(p,\mathbf{w})}{\partial p} & \frac{\partial y(p,\mathbf{w})}{\partial w_i} & \cdots & \frac{\partial y(p,\mathbf{w})}{\partial w_n} \\ -\frac{\partial x_1(p,\mathbf{w})}{\partial p} & -\frac{\partial x_1(p,\mathbf{w})}{\partial w_1} & \cdots & -\frac{\partial x_1(p,\mathbf{w})}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial x_n(p,\mathbf{w})}{\partial p} & -\frac{\partial x_n(p,\mathbf{w})}{\partial w_1} & \cdots & -\frac{\partial x_n(p,\mathbf{w})}{\partial w_n} \end{bmatrix}$$

In particular, $\frac{\partial y(p,\mathbf{w})}{\partial p} \geq 0$ and $\frac{\partial x_i(p,\mathbf{w})}{\partial w_i} \leq 0$, $\forall i$.

[18]Another way of thinking about this lemma (focusing on the y part of it), is "the change in profit from a change in price is proportional to the quantity produced".

So, if a firm produces y with cost \$1, and sells it for \$2, then producing 10 units, it obtains $\pi = 20 - 10 = 10$.

Now, if the firm increases the price of y to \$3, then it obtains $\pi = 30 - 10 = 20$.

The change in price leads to a change in profit of +10, which is exactly the amount of y being produced.

13 Class 13

13.1 Short-Run Profit Function (JR, chap. 3, theor. 3.4, p. 152)

The difference here, like in the short run cost function, is that some inputs (and its prices) are fixed.

$$\pi(p, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}}) \equiv \max_{y,\mathbf{x}} \{p \cdot y - \mathbf{w}'\mathbf{x} - \bar{\mathbf{w}}'\bar{\mathbf{x}}\} \quad s.t. \quad y = f(\mathbf{x}, \bar{\mathbf{x}}).$$

The solutions to this max problem are

$$y(p, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}}) \quad \text{and} \quad \mathbf{x}(p, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}}),$$

which are called the short-run, or restricted, output supply and input demand functions, respectively. We can write the short-run profit max problem also as

$$\pi(p, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}}) = \max_{y} p \cdot y - sc(y, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}}),$$

where the FOC is

$$\frac{\partial sc(y, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}})}{\partial y} = \text{short-run marginal cost} = p.$$

We can always write $sc(y, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}}) = VC(y) + FC = sc(y, \mathbf{w}) + FC(\bar{\mathbf{w}}, \bar{\mathbf{x}})$, where $VC(\cdot)$ is the variable cost, $FC(\cdot)$ is the fixed cost, and the arguments of $FC(\cdot)$, $(\bar{\mathbf{w}}, \bar{\mathbf{x}})$, are actually $(\bar{\mathbf{w}}'\bar{\mathbf{x}})$.

Now, let's say that y^1 maximizes short-run profits. Then,

$$p = \frac{\partial sc(y^1, \mathbf{w}, \bar{\mathbf{w}}, \bar{\mathbf{x}})}{\partial y} = \frac{VC(y^1)}{\partial y}.$$

Is it always the case that the firm wants to produce y^1 in the short-run? We have that

$$p \cdot y^1 - VC(y^1) - FC \geq -FC,$$

so the firm only produces y^1 if $p \geq \frac{VC(y^1)}{y^1} \equiv$ Avg Var Cost.

13.2 Partial Equilibrium under Perfect Competition (JR, chap. 4 (4.1), p. 165)

When dealing with equilibrium, it's necessary to have aggregate supply and demand functions. We'll focus on a single market and on the price of a single good (that's why "partial" equilibrium).

Aggregate demand:

$$q^d = q^d(p) = \sum_i q_i(p, y_i),$$

where we have $i = 1, \ldots, I$ individuals in society, and y_i represents the income of the i-th individual.

Aggregate supply:

$$q^s = q^s(p) = \sum_j q_j(p, \mathbf{w}),$$

where we have $j = 1, \ldots, J$ firms in the market.

The equilibrium conditions are as follows

- Number of firms is fixed: short run or some constraint to market entry;
- Free entry and zero profits: "long-run" equilibrium condition.

Example: constant returns to scale with limited capacity. This implies the following characterization:

- $MgC \equiv c$, it's like an unitary production cost;

- Max capacity = $q^j_{\max} = k$, for all firm j;
- Market demand = $q^d(c)$

Case 1: fixed number of firms, J, with $q^d(c) > k \cdot J$. This is the condition for positive profit. Recall that $q^d(\cdot)$ has as argument the price of the product. Since demand at price c (marginal cost) is bigger than the output of all industry, the actual price p (the one consumers are charged with) must be greater than c, in order to make $q^d(p) = k \cdot J$. Profit then is $q^d(p) - q^d(c)$.

Case 2: free entry and zero profits. Here, since the number of firms can increase indefinitely, it will do so until $q^d(c) = k \cdot J$. Thus, in this case, we have zero profit: $q^d(c) - q^d(c) = 0$. If all firms produce at maximum capacity, then the equilibrium number of firms $J^* = \frac{q^*}{k}$, where q^* is the amount of product demanded in the market.

Example: homogeneous workers, firms not using land. There is the following characterization:

- Profit function $\pi(w_i, r, A_i)$:
$$\pi_i = A_i F(K, L) - w_i L - rK,$$
with $i = \{1, 2\}$, A_i is the productivity of region i, and $A_1 > A_2$;
- Workers' mobility function $V(w_i, p_i)$ is function of wages (w_i) and rental prices (p_i, living costs) in each region;
- There is free mobility of workers: $V(w_i, p_i) = \bar{V}$, $\forall i$;
- Zero profit condition: $\pi(w_i, r, A_i) = 0$, $\forall i$.

In this case, we have that the region that pays higher wages will also have higher rental rates (it will be more expensive living there). In either regions, the condition of zero profit holds for the firms.

14 Class 14

14.1 Monopoly (TR, chap. 1, p. 65)

This part of the course is based on the first chapter of "The Theory of Industrial Organization", by Jean Tirole (JR). At the end of this document, there is a solved exercise of this chapter (I used the guide available in the book itself to solve it).

Single-product Monopolist

There is single producer who is aware of the effects of production on prices. The producer also knows the demand curve.

Here, the problem of the firm is

$$\max_{\{p,q\}} \{p \cdot q - C(q) \quad s.t. \quad q = D(p)\},$$

where q is the amount of the product produced by the monopoly, p is the price of this product, $C(q)$ is the cost of producing q units of this good, and $D(q)$ is the market demand for the good. We can rewrite the problem as

$$\max_{p} \{p \cdot D(p) - C[D(p)]\},$$

with the following first-order conditions (FOCs):

$$D(p) + p \cdot D'(p) - C'(q) \cdot D'(p) = 0$$
$$- C'(q) + p = -\frac{D(p)}{D'(p)} \longrightarrow \text{divide all by p} \longrightarrow$$
$$\longrightarrow \frac{p - C'(q)}{p} = -\frac{D(p)}{D'(p)} \cdot \frac{1}{p} = \frac{1}{\varepsilon},$$

where $\varepsilon = -\frac{D'(p)}{D(p)} \cdot p$. When $\frac{p-C'(q)}{p} \neq 0$, we know that there's some market power for this firm. This last expression is called mark-up, and ε is the price-elasticity of demand.

An alternative formulation in terms of q would be achieved by defining the inverse function $p = D^{-1}(q) = p(q)$. And then,

$$\max_{q}\{p(q) \cdot q - C(q)\}$$
$$\text{FOC:} \quad \underbrace{p'(q) \cdot q + p(q)}_{MgR = \text{Mg Revenue}} = \underbrace{C'(q)}_{MgC = \text{Mg Cost}}$$

Note that

$$MgR = \frac{\partial p}{\partial q} \cdot q + p = p\left(\frac{\partial p}{\partial q} \cdot \frac{q}{p} + 1\right) = p\left(1 - \frac{1}{\varepsilon}\right),$$

where $\frac{\partial p}{\partial q} \cdot \frac{q}{p} = -\frac{1}{\varepsilon}$.

Further, we note that monopolists never operates on an inelastic ($\varepsilon < 1$) portion of the demand curve. In this case, it's always better to reduce q and increase p. If $\varepsilon < 1$, then $MgR < 0$, which is not necessarily bad if the monopoly is able to reduce q; if it can reduce q, it will do so to avoid losing revenue.

Statement: prices are increasing in marginal cost under monopoly[19]. This follows from the fact that the monopolist knows everything about the market, so if he faces a higher MgC, he has (full) market power to increase p at his own will, in order to not lose revenue.

Multiproduct Monopolist

Now, we have vectors of prices and products, $\mathbf{p} = (p_1, \ldots, p_n)$; $\mathbf{q} = (q_1, \ldots, q_n)$ respectively. Demand and costs could be inter-related.

$$\pi = \sum_i p_i D_i(\mathbf{p}) - \underbrace{C(D_1(p_1), \ldots, D_n(p_n))}_{\text{Total Cost}}$$
$$\text{FOCs: } D_i(\mathbf{p}) + \sum_j p_j \cdot \frac{\partial D_j(\mathbf{p})}{\partial p_i} = \sum_j \frac{\partial C}{\partial q_j} \cdot \frac{\partial D_j(\mathbf{p})}{\partial p_i}, \ \forall i,$$

where i is the $i-$th product. Here are some examples.

- Intertemporal pricing (fad, habit formation): goods in different moments in time: $q_1 = D_1(p_1)$, $q_2 = D_2(p_2, p_1)$, $C_1(q_1)$ and $C_2(q_2)$, $\frac{\partial D_2}{p_1} < 0$, implying that more consumption in period 1 decreases consumption in period 2. Following TR (p. 71), the profit max problem is

$$\max_{\{p_1, p_2\}} \ p_1 D_1(p_1) - C_1(D_1(p_1)) + \delta(p_2 D_2(p_2, p_1) - C_2(D_2(p_2, p_1))),$$

[19]The proof of this statement is available at the class notes, but the intuition is the one explained here. Moreover, to grasp the proof, just note that the argument is about $c_2'(q)$ being greater than $c_1'(q)$ for all q, and by expressing that difference as an integral, we're implicitly assuming that $c_2'(q) < c_1'(q)$ for that particular q, which is not the case. Think about the x axis: if $c_2'(q) > c_1'(q) \, \forall q$, then the integral should be $[c_1'(x) - c_1'(x)]$.

where δ is the discount factor. Note that each good is like a single good in each period. Thus, letting $D_2 = \delta D_2$ and $C_2 = \delta C_2$, we get that the first-order conditions are

$$\begin{aligned}
(p_2): \quad & p_2 \cdot \frac{\partial D_2}{\partial p_2} + D_2(p_2, p_1) - \frac{\partial C_2}{\partial q_2} \cdot \frac{\partial D_2}{\partial p_2} = 0 \\
& p_2 \cdot \frac{\partial D_2}{\partial p_2} + D_2(p_2, p_1) = \frac{\partial C_2}{\partial q_2} \cdot \frac{\partial D_2}{\partial p_2} \\
& \frac{D_2(p_2, p_1)}{\partial D_2 / \partial p_2} + p_2 = \frac{\partial C_2}{\partial q_2}
\end{aligned}$$

$$\begin{aligned}
(p_1): \quad & \underbrace{p_1 \frac{\partial D_1}{\partial p_1} + D_1(p_1) - \frac{\partial C_1}{\partial q_1} \cdot \frac{\partial D_1}{\partial p_1}}_{\text{period 1}} + \underbrace{p_2 \cdot \frac{\partial D_2}{\partial p_1} - \frac{\partial C_2}{\partial q_2} \cdot \frac{\partial D_2}{\partial p_1}}_{\text{period 2}} = 0 \\
& D_1(p_1) + p_1 \cdot \frac{\partial D_1(p_1)}{\partial p_1} = \frac{\partial C_1}{\partial q_1} \cdot \frac{\partial D_1(p_1)}{\partial p_1} + \underbrace{\frac{\partial C_2}{\partial q_2} \cdot \frac{\partial D_2}{\partial p_1} - p_2 \cdot \frac{\partial D_2}{\partial p_1}}_{= D_2 * D_2^{'} \text{ ratio (?)}} \\
& \underbrace{D_1(p_1) + p_1 \cdot \frac{\partial D_1(p_1)}{\partial p_1}}_{\text{decreasing on prices}} = \frac{\partial C_1}{\partial q_1} \cdot \frac{\partial D_1(p_1)}{\partial p_1} + \underbrace{D_2(p_2, p_1) \cdot \frac{\partial D_2(p_2, p_1)/\partial p_1}{\partial D_2(p_2, p_1)/\partial p_2}}_{(*)},
\end{aligned}$$

where LHS[20] being "decreasing on prices" means that by adding $(*)$, we have a lower LHS. This is because p_1 affects not only consumption in period 1, but also in period 2.
Moreover, second-order conditions implies that if the LHS is decreasing on p, then p_1 will be lower than under the "traditional"[21] setting;

- Learning-by-doing: it's what happens in some industries where cost reductions are achieved over time simply because of learning. In this case, we have that $q_1 = D_1(p_1)$, $q_2 = D_2(p_2)$, $C_1(q_1)$, $C_2(q_2, q_1)$, $\frac{\partial C_2}{\partial q_1} < 0$. Again, we are considering two periods of production (it could be one or two products, it doesn't matter because product(s) are single in each period). The profit max problem of the monopolist is

$$\max_{\{p_1, p_2\}} \; p_1 D_1(p_1) - C_1(D_1(p_1)) + \delta(p_2 D_2(p_2) - C_2(D_2(p_2), D_1(p_1))).$$

Thus, the first-order conditions, assuming $D_2 = \delta D_2$ and $C_2 = \delta C_2$, are

$$\begin{aligned}
(p_1): \quad & D_1(p_1) + p_1 \cdot \frac{\partial D_1(p_1)}{\partial p_1} - \frac{\partial C_1(D_1(p_1))}{\partial q_1} \cdot \frac{\partial D_1(p_1)}{\partial p_1} - \frac{\partial C_2(D_2(p_2), D_1(p_1))}{\partial q_1} \cdot \frac{\partial D_1(p_1)}{\partial p_1} = 0 \\
& D_1(p_1) + p_1 \cdot \frac{\partial D_1(p_1)}{\partial p_1} = \frac{\partial C_1(D_1(p_1))}{\partial q_1} \cdot \frac{\partial D_1(p_1)}{\partial p_1} + \underbrace{\frac{\partial C_2(D_2(p_2), D_1(p_1))}{\partial q_1}}_{\text{¡0}} \cdot \underbrace{\frac{\partial D_1(p_1)}{\partial p_1}}_{\text{¡0}}
\end{aligned}$$

$$\begin{aligned}
(p_2): \quad & D_2(p_2) + p_2 \cdot \frac{\partial D_2(p_2)}{\partial p_2} - \frac{\partial C_2(D_2(p_2), D_1(p_1))}{\partial q_2} \cdot \frac{\partial D_2(p_2)}{\partial p_2} = 0 \\
& D_2(p_2) + p_2 \cdot \frac{\partial D_2(p_2)}{\partial p_2} = \frac{\partial C_2(D_2(p_2), D_1(p_1))}{\partial q_2} \cdot \frac{\partial D_2(p_2)}{\partial p_2}.
\end{aligned}$$

Note in the FOC related to (p_1) that if $\frac{\partial C_2}{\partial q_1} \cdot \frac{\partial D_1}{\partial p_1} > 0$, then p_1 will be smaller than in the "traditional" case, since if the monopolist increases p_1, the costs in period 2, C_2, also increases.

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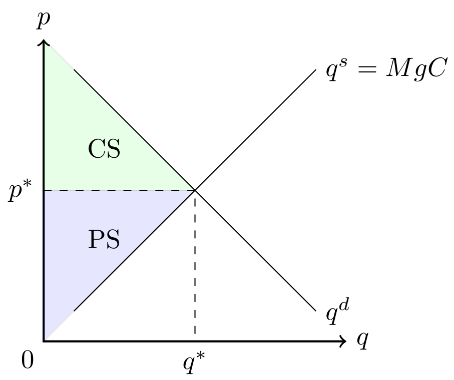
[20]The (?) sign in the second step is because prof Rodrigo just wrote the third step. I tried to write the previous steps and, in order to get according to his notes, I deliberately assumed that expression in the second step as equal to $(*)$.

[21]Traditional meaning the case where there is no intertemporal dependency on prices charged by the monopolist.

14.2 Monopoly and Welfare (TR, chap. 1)

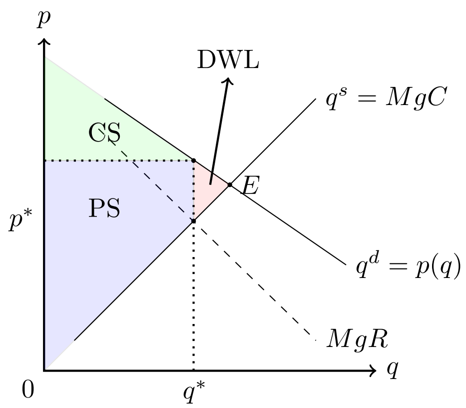
The idea here is to analyze what happens in the society in terms of welfare when we don't have a situation like perfect competition. In fact, real economy is made mostly by markets that do not operate under perfect competition.

Under perfect competition, there is both consumer surplus (CS) and producer surplus (PS), and that's all. Importantly, we note that the supply curve q^s equals the marginal cost MgC, since it's perfect competition.



The total gains from trade in this situation are given by $PS + CS$.

Under monopoly, we have smaller PS and CS, because now there's also a dead-weight loss (DWL), an inefficiency created by the existance of the monopoly.



We note in this graph that the DWL region is not "gained" by anyone, nor producer of consumer, i.e., it's a loss for society as a whole. We also see that the total surplus will be smaller, once there's a reduction of CS, even though there is an increase of PS, i.e., an increase in firm's profits. From TR, we note that decrease in total surplus exceeds the increase in profit by an amount equal to the dead-weight loss. Recall that under perfect competition, there is zero profit for all firms, and thus the DWL doesn't exist.

Moreover, we note that the equilibrium under a monopolized market will never be in point E, which is the equilibrium of perfect competition.

15 Class 15

15.1 Monopoly and Welfare (cont'd)

Using the last graph from previous section, we can analyze the problem of rent-seeking. Rent-seeking happens when a firm is willing to spend up to the whole PS area in money, in order to keep it's position as monopolist. It means that the firm will use money in an inefficient way just to keep it's monopoly power, which is socially inefficient itself because of DWL.

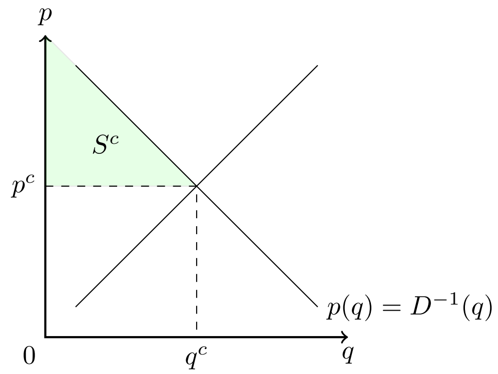
15.2 Price Discrimination (TR, chap. 3, p. 133)

We will discuss the types of price discrimination that a monopolist may practice in the market.

15.2.1 Perfect Discrimination (1st Degree Price Discrimination)

In this scenario, consumers have a willingness to pay given by V_i, and the monopolist knows it, so it charges $p_i = V_i$ of each consumer. That's the key feature of this formulation: monopolist charges each one at the highest price possible. An alternative setting would be: consumers have identical demand $q_i = \frac{D(p)}{n}$, with n consumers. Thus, the aggregate demand is $q = \sum_i q_i = D(p)$.

Assume that monopolist charges a "two-part" tariff: $T(q) = A + p \cdot q$, where A is exactly the equal "share" each consumer pays for it's own demand (it may be a fixed premium, if all consumers are homogeneous, or just an amount correspondent to the consumer's willingness to pay, if they're heterogeneous). We would have the following equilibrium



where S^c is the consumer surplus, and thus the individual consumer surplus is $\frac{S^c}{n}$. Thus, with many consumers ($n \to \infty$), $S^c = 0$, i.e., the monopolist takes all S^c. If marginal price is p^c (competitive price), then the consumer surplus is given by:

$$S^c = \int_0^{q^c} [p(q) - p^c]\, dq.$$

The monopolist can set $A = \frac{S^c}{n}$ and $p = p^c$, and then the two-part tariff would become:

$$T(q_i) = \begin{cases} \frac{S^c}{n} + p^c \cdot q_i \ , & \text{if } q_i > 0 \\ 0 \ , & \text{if } q_i = 0 \end{cases}$$

and thus $S^c = 0$ both because $S^c \to 0$ as $n \to \infty$, and because $p(q) - p^c = 0$ inside the integral. And note that the only way the consumer would not be charged so heavily is by consuming nothing, $q_i = 0$.

Demand will be q^c, and profits will be

$$\pi = p^c \cdot q^c + S^c - c(q^c),$$

and we notice again that all S^c is now part of the monopolist's profit.

15.2.2 Discrimination in Different Markets (3rd Degree Price Discrimination)

Here, the monopolist produces a single good with cost $c(q)$, but sells it in m different markets, each one with demand $D_i(p_i)$, $i = 1, \dots, m$. Thus, the profit max problem for the monopolist is

$$\max_{\{p_i\}, \forall i} \pi = \sum_{i=1}^{m} p_i \cdot D_i(p_i) - c\left(\sum_{i=1}^{m} D_i(p_i)\right).$$

which means that the monopolist must maximize each i price in the m different markets, considering the cost of producing the same good in amount equivalent to the sum of all m demands. The first-order conditions of this problem are

$$D_i(p_i) + p_i \cdot D_i'(p_i) - c'(q) \cdot D_i'(p_i) = 0$$
$$\frac{p_i - c'(q)}{p_i} = -\frac{D_i(p_i)}{D_i'(p_i) \cdot p_i}$$
$$\frac{p_i - c'(q)}{p_i} = \frac{1}{\varepsilon_i},$$

where $\varepsilon_i = -\frac{D_i'(p_i) \cdot p_i}{D_i(p_i)}$ is the elasticity of demand in market i. Optimal pricing implies that the monopolist should charge more in markets with the lower elasticity of demand[22].

15.2.3 Discrimination Through Screening (2nd Degree Price Discrimination)

Prof. Rodrigo says it's the most important type of price discrimination. The idea here is that the monopolist knows that customers are heterogeneous, but differently from 3rd deg. disc., here there's no exogenous signal of each consumer's demand function (such as age or occupation). Then, the task is to offer different bundles (price and quantity, price and quality, etc.) to different customers, in order to achieve perfect discrimination.

We start with a two-part tariff system again, which generally is not optimal, and then we shall proceed to a consideration of more general non-linear pricing schemes. Consider $T(q) = A + p \cdot q$. Also, suppose that consumers have the following preferences:

$$u_i = \begin{cases} \theta_i V(q) - T\ , & \text{if buys } q \text{ and pays } T \\ 0\ , & \text{if doesn't buy} \end{cases}$$

where $V(0) = 0$, $V'(q) > 0$, $V''(q) < 0$, i.e., $V(\cdot)$ is a increasing and concave function, and there's a decreasing marginal utility of consumption in this representation of the utility function. θ_i is a taste parameter of the consumer i, while $V(\cdot)$ is the same for all consumers.

Assume that there are two groups of consumers. Those with taste parameter θ_1 are in proportion λ; those with taste parameter θ_2 are in proportion (1-λ). Also, $\theta_2 > \theta_1$, and the monopolist produces at a constant marginal cost $c < \theta_1 < \theta_2$. Finally, assume that $V(q) = \frac{1-(1-q)^2}{2}$, so that $V'(q) = (1 - q)$ is linear in quantity.

If the consumer buys the good at price p (ignoring A), demand is determined from the following consumer's max problem

$$\max_{q}\ \theta_i V(q) - p \cdot q,$$

[22] From TR: "This rule explains why students and senior citizens are given discounts by private firms with no redistribution intention, why legal and medical services are priced according to the customer's income or amount of insurance coverage, why the prices of goods in different countries sometimes do not reflect transportation costs and import taxes, and why first-time subscribers to a magazine are given discounts".

which yields in

$$\theta_i V'(q) = p.$$

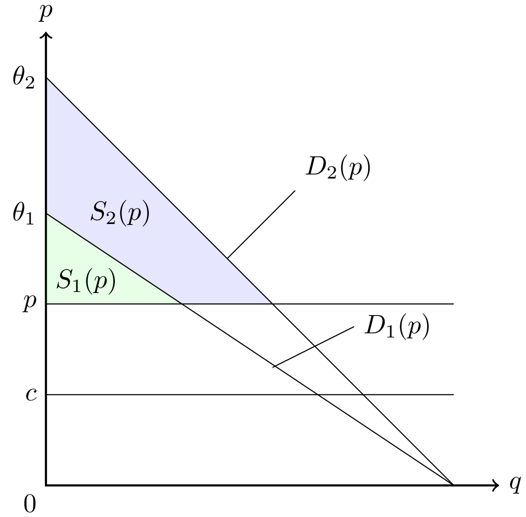
By our specification of $V'(q)$, we then have that

$$q = 1 - \frac{p}{\theta_i} \Rightarrow D_i(p) = 1 - \frac{p}{\theta_i}.$$

The net consumer surplus, with $A = 0$, is

$$\begin{aligned} S_i(p) &= \theta_i \left[\frac{1-(1-q)^2}{2} \right] - p \cdot q \\ &= \frac{(\theta_i - p)^2}{2 \cdot \theta_i}. \end{aligned}$$

Note that $S_i(\theta_i) = 0$, and that the surplus is always higher for θ_2 types. The demand curves and the net surpluses are depicted in the figure below.



Now, define θ as the harmonic mean of θ_1 and θ_2:

$$\frac{1}{\theta} \equiv \frac{\lambda}{\theta_1} + \frac{1-\lambda}{\theta_2}.$$

So the aggregate demand will be

$$D(p) = \lambda \cdot D_1(p) + (1-\lambda) \cdot D_2(p) = 1 - \frac{p}{\theta}.$$

Perfect Discrimination

Thinking about perfect discrimination under the 2nd degree price discrimination, we first need that the monopolist observes θ_i, i.e., it can differentiate among consumers. The monopolist can charge marginal price $p_1 = c$ and demand a personalized fixed premium equal to each consumer's net surplus at price c. For consumer i ($i = 1, 2$), the fixed premium, A_i, is

$$A_i = S_i = \frac{(\theta_i - c)^2}{2 \cdot \theta_i} \equiv \pi_1,$$

and we note that the fixed premium is naturally higher for the high-demand consumer. Prof Rodrigo calls this expression as π_1, which is the highest profit level under the 2nd deg. price disc., because the monopolist is discriminating everyone.

Actually, the monopolist's profit in this scenario would be

$$\pi_1 = \lambda \cdot \frac{(\theta_1 - c)^2}{2 \cdot \theta_1} + (1 - \lambda) \cdot \frac{(\theta_2 - c)^2}{2 \cdot \theta_2},$$

which is just the previous expression weighted by the proportion of each type of consumer.

"Simple" Monopoly Pricing

Here, we suppose that there's full arbitrage between consumers, so that the monopolist is forced to charge a fully linear tariff: $T(q) = p \cdot q$. The profit max problem under this new price p_2 (or p^m) is

$$\max_p \ \pi = (p - c) \cdot D(p) = (p - c) \cdot \left(1 - \frac{p}{\theta}\right),$$

which yields in the monopoly price

$$p_2 = \frac{c + \theta}{2},$$

and the monopoly profits are

$$\pi_2 = \frac{(\theta - c)^2}{4\theta}.$$

Note that these computations assume that the monopolist wants to serve both types of costumers. Another strategy might be to serve only type-θ_2 consumers. Such strategy will be optimal if the monopoly price relative to this category (which is $(c + \theta_2)/2$) exceeds θ_1, and the fraction of type-θ_1 consumers is sufficiently small.

Two-Part Tariff

Again, monopolist serves both types of consumers, and now uses discrimination through screening using the two-part tariff.

If the linear price is p, from the previous graph, then type-θ_1 individuals will be willing to pay up to $S_1(p) = \frac{(\theta_1 - p)^2}{2\theta_1}$ to participate in this market. And if type-θ_1's participate, so do type-θ_2's, since $\theta_2 > \theta_1 \Rightarrow S_2(p) > S_1(p) = A$, where A is the highest fixed fee the monopolist can charge to serve all consumers. Thus, the monopolist maximizes

$$\max \ \pi_3 = \max_p \ A + p \cdot q - c \cdot q = \max_p \ S_1(p) + (p - c) \cdot D(p) =$$

$$= \max_p \ \frac{(\theta_1 - p)^2}{2\theta_1} + (p - c) \cdot \left(1 - \frac{p}{\theta}\right),$$

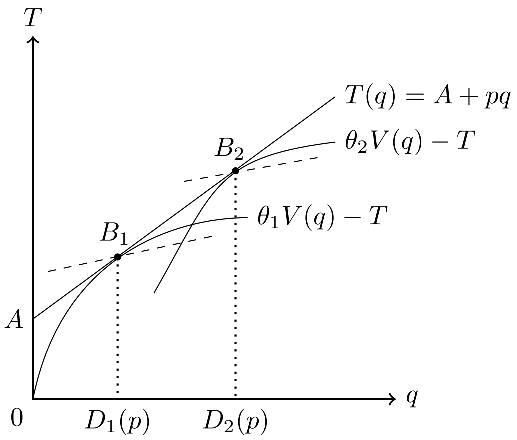
which yields in the price

$$p_3 = \frac{c}{2 - \theta/\theta_1}.$$

We end up with the following relation: $\pi_1 \geq \pi_3 \geq \pi_2$, which follows from the analysis of the prices (under the hypothesis that all types of consumers are served):

$$p_1 = c < p_3 < p2 = p^m.$$

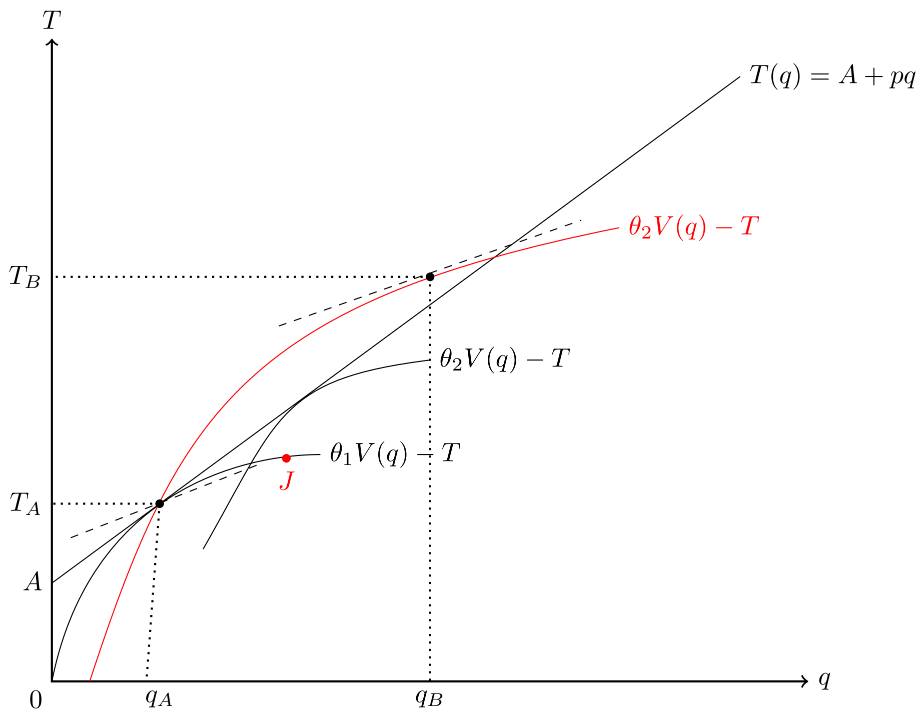
Thus, the marginal price is intermediate between the competitive price (p_1, which would be paid by the marginal consumer under perfect discrimination) and the monopoly price. Moreover, we can think that π_2 is the lowest profit level because in this case the monopolist has fewer instruments to extract social surplus than in π_3: in π_3, the monopolist has A and p, while in π_2 he has only p.



The graph above depicts the (q, T) space, a representation of the two-part tariff scheme. The straight line represents the optimal two-part tariff, $T(q) = A + pq$. The two types of consumers' indifference curves are concave, because $V(q)$ is. Because $\theta_2 > \theta_1$, the type-θ_2 consumers' indifference curve is steeper than the type-θ_1 consumers' when the curves cross. Under the optimal two-part tariff, type-θ_1 consumers pick the bundle B_1, while type-θ_2 consumers pick B_2. By construction, low-demand consumers have no net surplus (their indifference curve goes through the origin, they have the same utility with B_1 and nothing at all), whereas high-demand consumers have positive net surplus.

The dashed lines are the indifference curves for the monopolist, given by $T - cq$ (constant), and are called isoprofits. Because $c < p$, these indifference curves are flatter than the optimal two-part tariff.

What we learn from this graph is that the monopolist doesn't extract all surplus from type-θ_2 consumers because of the type-θ_1's. If the monopolist chooses the highest isoprofit curve (the one tangent to $\theta_2 V(q) - T$ up there), type-θ_1 consumers will choose to buy nothing, and then the monopoly will extract everything from type-θ_2 guys. The solution for the monopolist would be something like the graph below.



Here, the firm offers both bundles q_A and q_B in such a way that is maximizes profit with type-θ_2 guys by extracting all of their surplus; note that the isoprofit in point (q_B, T_B) is tangent to the indifference curve of the type-θ_2 consumers.

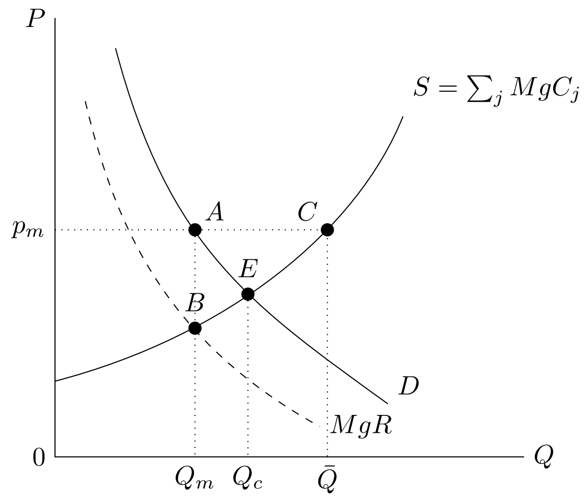
The monopolist is choosing to do so because it's better to extract all surplus from type-θ_2 consumers than from type-θ_1's. The reason is that, in order to extract all surplus from type-θ_1's, the monopolist would need to offer the bundle J, but then the type-θ_2 consumers would also prefer to consume J, since it is in a higher (more at right, more distant from the origin) indifference curve for them. Hence, the optimal choice is to offer q_A for type-θ_1 consumers, at the cost of not extracting all their surplus; and offer q_B for type-θ_2's, and extract all of their surplus.

The idea is that it's better to distort the choice of type-θ_1's, offering them a very low q_A, and offer a high q_B for type-θ_2's and extract all of them. Think about the aviation industry: economy-class tickets are so bad that rich customers (type-θ_2 customers) not even dream about buying it, and thus they go straight to the first-class (q_B), even if it's much more expensive.

16 Class 16

16.1 Collusion (*Conluio*, Cartels) (TR, ch. 6, p. 239; JR, ch. 4, p. 170)

We start considering J identical firms. When they agree with a collusion production, we get that each firm produces $q_j = \frac{Q_m}{J}$, with market price p_m. One key point here is that, under p_m, we have that $MgR = MgC$ (point B), but if one firm deviates from producing Q_m, it would profit more by selling at lower prices.



Graphically, we check that the equilibrium under collusion is unstable, i.e., there is incentive to deviate from producing $q_j = \frac{Q_m}{J}$ per firm. That's because by producing this amount, each firm is actually at point B, which doesn't maximize its profits, once this happens only in point C, where $MgC_j = p_m$.[23] By producing Q_m, the collusion forces each member to have lower profits than what it could make by producing $\bar{Q}$. Moreover, we observe that any of these situations, whether B (charging p_m there in point A) or C, is good for consumers; the only case where consumers are fully satisfied is at the point of equilibrium E, where prices are actually lower than p_m.

Thus, if firms take p_m as given, they want to set $MgC_j = p_m$, which leads to $\bar{Q}$ aggregate production. We have that each firm's profit is a function of the amount produced by all firms in the cartel:

$$\Pi^j = \Pi^j(q^1, \ldots, q^j, \ldots, q^J) \quad \longrightarrow \quad \frac{\partial \Pi^j}{\partial q^i} < 0 \,, i \neq j$$

[23] Of course, assuming that prices would continue to be p_m (higher than the equilibrium price) even with this deviant firm operating out of the collusion agreement.

$$\text{(Each firm in the collusion wants to) } \max \sum_{j=1}^{J} \Pi^j \quad \longrightarrow \quad \text{FOCs: } \underbrace{\frac{\partial \Pi^k}{\partial q^k}}_{>0} + \underbrace{\sum_{j \neq k} \frac{\partial \Pi^j}{\partial q^k}}_{<0} = 0 \;, \forall k.$$

This FOC says that each firm's profit is increasing in its own output at Q_m, meaning that each can increase its own profit by increasing output away from its assignment under Q_m — provided, of course, that everyone else continues to produce their assignment under Q_m! Note that when one firm deviates from the agreement of producing Q_m, it will increase profits because it would be able to sell positive amounts of output charging a price lower than p_m; this continues to happen until the total output (this deviant firm plus all others) reaches $\bar{Q}$, where prices are again p_m. But if even one firm succumbs to this temptation, Q_m will not be the output vector that prevails in the market. Thus, all firms will produce more, charge less, until total output is $\bar{Q}$.

16.2 Oligopoly - Cournot (TR, ch. 5, p. 218; JR, ch. 4, p. 174)

Cournot's approach is about (oligopolistic) competition in quantity. The traditional case is that in which firms takes the production of competitors as given (which is not very realistic), and they consider this information when maximizing their own profit. Moreover,

- Each firm has the same cost function: $c(q^j) = c \cdot q^j \;, j = 1, \ldots, J$;
- There's a single market, and all firms face the same (inverse) demand curve: $p = a - b \cdot \sum_j q^j \;, a > c$.

Cournot-Nash Equilibrium

The problem of a j-th firm in a Cournot Oligopoly is

$$\max_{q^j} \; \Pi(\bar{q}^1, \ldots, q^j, \ldots, \bar{q}^J) = (a - b \cdot \sum_{k=1}^{J} q^k) \cdot q^j - c \cdot q^j$$

The solution to the Cournot's oligopoly is called Cournot-Nash Equilibrium. To derive it, we start with the first-order conditions of the maximization problem:

$$a - 2b \cdot q^j - b \cdot \sum_{k \neq j} \bar{q}^k - c = 0.$$

In a market equilibrium, this condition holds for all J firms. The solution in Cournot is symmetric, with all firms behaving the same way

$$q^j = \bar{q} = \frac{a - c}{b(J+1)} \;, \forall j,$$

$$\bar{p} = a - \frac{J(a-c)}{J+1} \;,$$

$$\Pi = \frac{(a-c)^2}{(J+1)^2 \cdot b},$$

from which follows directly that

$$\bar{p} - c = \frac{a-c}{J+1} \longrightarrow \lim_{J \to \infty} (\bar{p} - c) = 0.$$

Thus, with only 1 firm, Cournot's model is a monopoly. With $J \to \infty$, Cournot's model is a perfect competition model.

16.3 Oligopoly - Bertrand (TR, ch. 5, p. 209; JR, ch. 4, p. 175)

Bertrand's model is about (oligopolistic) competition in prices. This means that each firm takes competitors' prices as given when maximizing their own profits, which is more realistic. We will consider the simplest case, a duopoly.

Both firms face the same demand curve: $Q = \alpha - \beta p$, where p is the market price. Also, both firms are homogeneous, and they have constant marginal costs. Once consumers are always looking for the smallest price, we have that,

- when both firms charge the same price, they split the market equally;
- when their prices differ, the firm charging a smaller price takes the whole market, supposing it will be able to attend all demand.

Here, each firm's profit clearly depends on its rival's price as well as its own. Taking firm 1 for example, for all non-negative prices below α/β (the price at which market demand is zero), profit will be

$$\Pi^1(p^1, p^2) = \begin{cases} (p^1 - c)(\alpha - \beta p^1), & c < p^1 < p^2, \\ \frac{1}{2}(p^1 - c)(\alpha - \beta p^1), & c < p^1 = p^2; \\ 0, & \text{otherwise.} \end{cases}$$

Note that firm 1's profit is positive as long as its price exceeds marginal cost. Other things being equal, it will be largest, of course, if firm 1 has the lowest price, and only half as large if the two firms charge the same price. Its profit need never be negative, however, because the firm can always charge a price equal to marginal cost and assure itself zero profits at worst. The situation for firm 2 is symmetrical. Hence, we can suppose that each firm i restricts attention to prices $p^i \geq c$.

What is the Nash equilibrium in this case? As long as profits are positive, both firms have incentives to lower prices, because given the market demand (supposing we know it), whoever charges the lower price gets all market, and thus max profit.

Clearly, this is not an equilibrium. The only equilibrium is both firms charging $p^i = MgC$, which is the lowest possible price any firm can charge, and thus both firms would end up having zero profit.

16.4 General Equilibrium (JR, ch. 5, p. 195; MWG, ch. 15, p.)

The thing about general equilibrium is to check its existence, uniqueness, and stability. We start by the first one, and we'll introduce a very useful and popularized tool for this analysis: the Edgeworth Box.

We begin considering the simplest possible case, an endowment economy with two agents and two goods, defined by the following characteristics:

- agents: $i = 1, 2$;
- goods: $l = 1, 2$;
- i's consumption vector: $x_i = (x_{1i}, x_{2i})$;
- consumption set: $\mathbb{R}^2_+$;
- there are preferences $\succsim_i$ that order bundles in $\mathbb{R}^2_+$;
- i's endowment vector (MWG notation): $\omega_i = (\omega_{1i}, \omega_{2i})$;
- total endowment in the economy: $\bar{\omega}_l = \omega_{l1} + \omega_{l2}$.

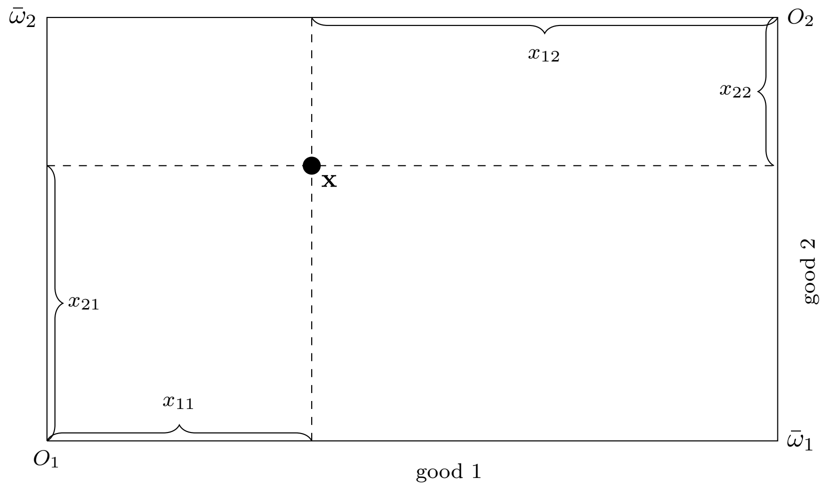
We will call an **allocation** any $\mathbf{x} \in \mathbb{R}^2_+$, which is a consumption vector (non-negative) for each agent:

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) = ((x_11, x_21), (x_12, x_22)).$$

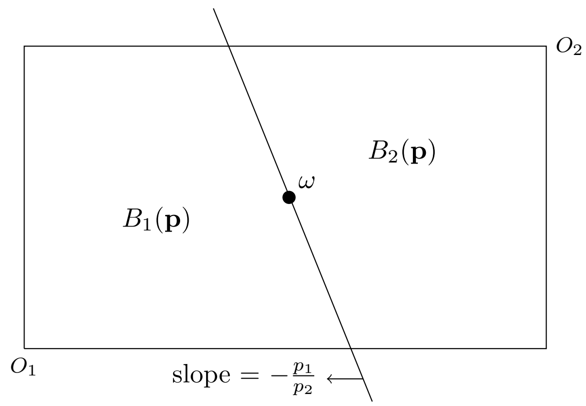
A **feasible** allocation is given by any $\mathbf{x} \in \mathbb{R}^2_+$ that satisfies the following condition:

$$\mathbf{x}_{l1} + \mathbf{x}_{l2} \leq \bar{\omega}_l, \; l = 1, 2.$$

This condition, when satisfied with equality, can be graphically represented in the Edgeworth box:

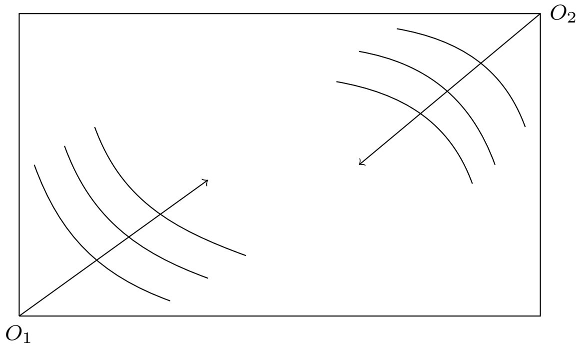


In this economy, wealth is determined from endowments and prices[24]. Actually, we can define a budget set for this economy: $B_i(\mathbf{p}) = \{\mathbf{x}_i \in \mathbb{R}^2_+ \mid \mathbf{p}'\mathbf{x} \leq \mathbf{p}'\omega_i\}$. This also can be represented in the Edgeworth box:



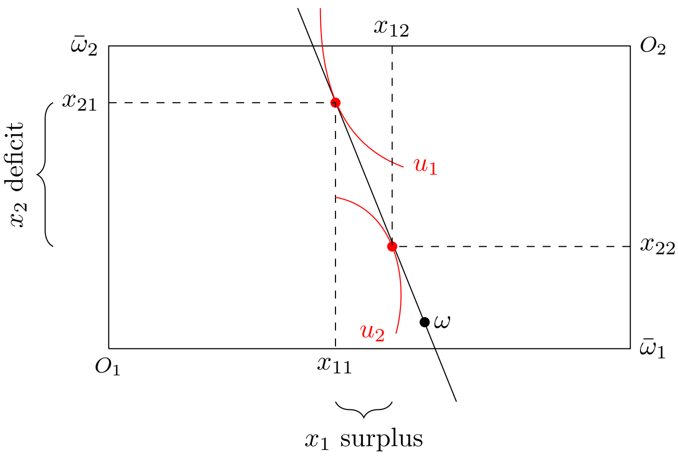
Preferences are convex, continuous, and monotonic:

[24]Note that prices will appear graphically as a straight line through the box.



What individuals actually do here? They exchange their endowments in order to have maximum utility, given the total endowment of each good in the economy. The example below is not an equilibrium, because:

- $x_{11} + x_{12} < \bar{\omega}_1$;
- $x_{21} + x_{22} > \bar{\omega}_2$.



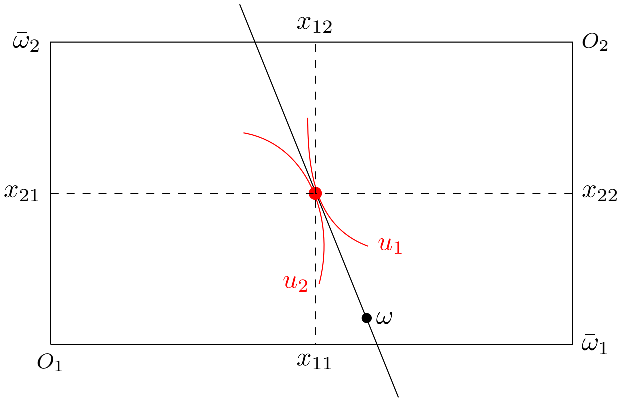
From this discussion, we can define the Walrasian Equilibrium.

Def.: A Walrasian (Competitive) Equilibrium is a price vector $\mathbf{p}^*$ and an allocation $\mathbf{x}^*$ such that:

- $\mathbf{x}_1^* \succsim \mathbf{x}_i'$, $\forall \mathbf{x}_i' \in B_i(\mathbf{p}^*)$, $\forall i$, meaning that the allocation $\mathbf{x}^*$ is at least as good as any other bundle in the budget set;
- $\sum_i \mathbf{x}_{li} = \bar{\omega}_l$, $\forall l$, meaning that the sum of each agent's consumption vector is equivalent to the total endowment of each good in the economy. This is also called "market clearing condition".

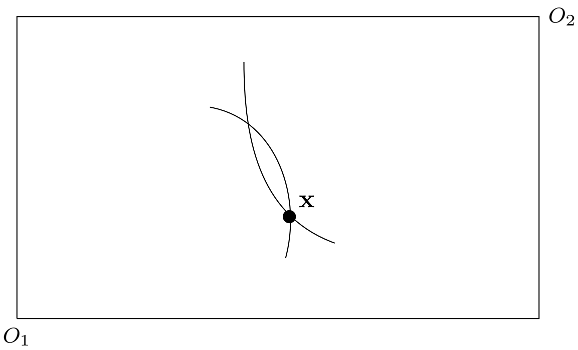
The following graph represents an equilibrium: there is no surplus nor deficit in the economy.[25]

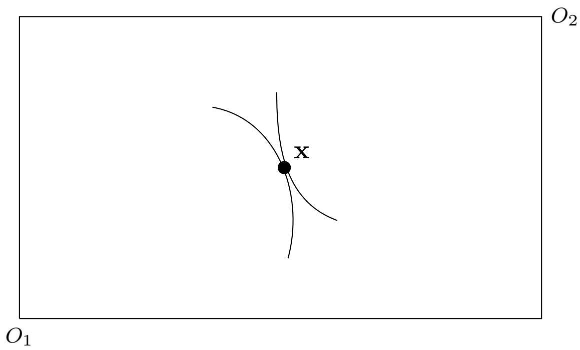
[25]There is a brief discussion in MWG about offer-curves, and Prof. Rodrigo also presents it in class. Due to ability limitations, I was not able to reproduce them here in LaTeX. So, check MWG, pp. 518-522, where these curves are depicted.



Def.: an allocation $\mathbf{x}$ is Pareto Optimal (Efficient), or just PO, if there is no other allocation $\mathbf{x}'$ such that $\mathbf{x}'_i \succsim \mathbf{x}_i$, $\forall i$, and $\mathbf{x}'_i \succ \mathbf{x}_i$ for some i.

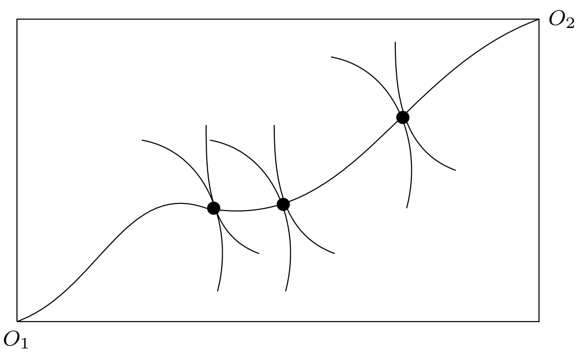
In the cases depicted below, the figure at left shows an allocation that is not PO, once any bundle "inside the eye-shaped form" between the indifference curves is strictly better than $\mathbf{x}$; the figure at right shows a PO allocation, once any other bundle other than $\mathbf{x}$ necessarily makes one of the agents worse off.

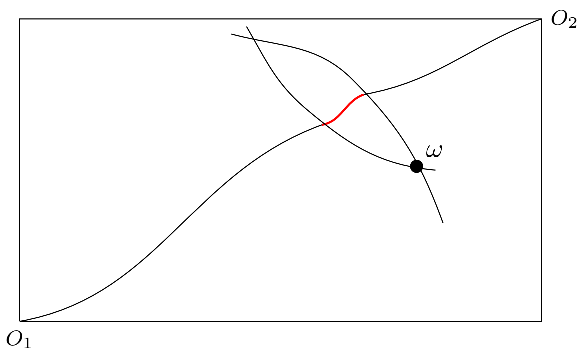




Def.: the Pareto Set is the set of all PO allocations. An example of a Pareto Set is depicted below at left.

Def.: the Contract Curve is the set of PO points within a Pareto Set that improve upon an initial endowment from the perspective of all consumers. An example of a Contract Curve is depicted below at right; the contract curve in this case is just the portion in red of the Pareto Set.





17 Class 17

17.1 General Equilibrium (JR, ch. 5, p. 195; MWG, ch. 15, p.) (cont'd)

Looking now to the operational details, we note first that efficiency in consumption is always given by the equality of MRS_i inside the Edgeworth box.

17.1.1 Economy with One Consumer and One Firm

Consider an economy with only one consumer and one firm, where both are price takers. In this economy, there are two goods: labor (z) and a consumption good (x_2) produced by the firm. The consumer has preferences $\succsim_i$ over the consumption good and leisure (x_1), such that his utility function is $u(x_1, x_2)$. Moreover, the consumer possesses an endowment $\bar{L}$ of time.

The firm has a production function defined by $f(z)$. Importantly, the consumer owns the firm and receives Π as a transfer.[26]

Hece, we have that the problem of the firm is

$$\max_{z} \; p \cdot f(z) - w \cdot z \quad \Rightarrow \quad \text{Sol.: } z(p,w),\; y(p,w),\; \Pi(p,w),$$

while the consumer's problem is

$$\max_{\{x_1,x_2\}} \; u(x_1,x_2) \quad s.t. \quad p \cdot x_2 = w(\bar{L} - x_1) + \Pi(p,w) \quad \Rightarrow \quad \text{Sol.: } x_1(p,w) \text{ and } x_2(p,w).$$

It follows directly that the Walrasian Equilibrium is given by p^* and w^*, such that

$$x_2(p^*,w^*) = y(p^*,w^*)$$
$$z(p^*,w^*) = \bar{L} - x_1(p^*,w^*).$$

These two conditions tells us that, in equilibrium, the consumption of the good x_2 is exactly equal to the consumer's income, which is actually the firm's profit; and the supply of labor z to the firm is exactly equal to the portion of $\bar{L}$ that the consumer decides not to have leisure (note the minus sign in x_1).

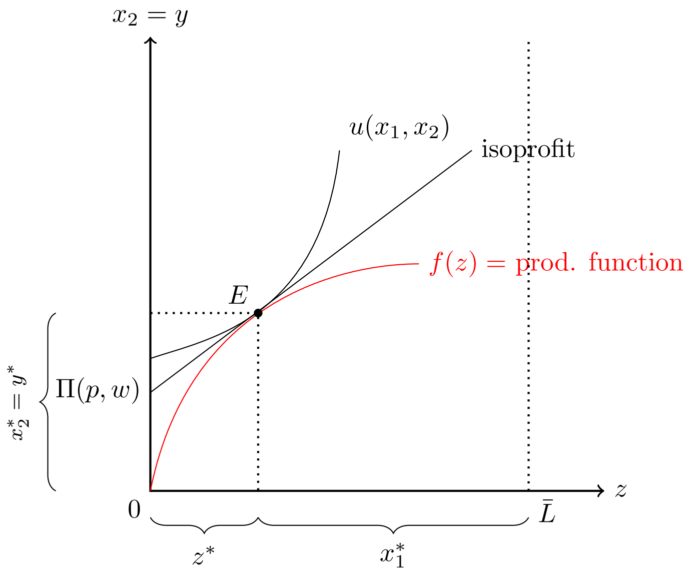
The first-order condition to both consumer and firm is:

$$\underbrace{\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \frac{u_1}{u_2} = \frac{w}{p}}_{\text{consumer}} \qquad \text{and} \qquad \underbrace{p \cdot f'(z) = w}_{\text{firm}}.$$

Notice that $f'(z)$ is the rate at which time is technically turned into x_2. Using both conditions, we have that

$$MRS = \frac{u_1}{u_2} = f'(z) = MRTS \quad \Leftrightarrow \quad MRS = MRTS.$$

The following graph depicts the equilibrium in this economy.



[26]This case where there's only one consumer and one firm, and the consumer is the owner of the firm, is also called Economy of Robinson Crusoe. For a complete discussion about it, see JR, p. 226, example 5.2.

Firstly, notice that the y axis refers to the amount of x_2 that will be consumed, which is the same as analyzing the income of the consumer, once in equilibrium, $x_2 = y$. In equilibrium, we have that the optimum amount of x_2^* is in point E.

The x axis refers to the supply of labor z. Once the consumer has a finite endowment of time ($\bar{L}$), the firm can only demand an amount of labor less than or equal to the whole time the consumer has available; but since the consumer finds utility in not working (leisure), we know that $z < \bar{L}$. The optimum share of time dedicated to labor is z^*, indicated by the equilibrium point E, and the remainder is leisure (x_1^*).

Analyzing the curves, we begin with the production function in red. The firm maximizes profit when this function is tangent to the isoprofit curve. This happens in point E. Importantly, if the firm demands more labor, it will have lower profit. And above all, notice that, in equilibrium E, the consumer's utility function $u(x_1, x_2)$ is also tangent to the isoprofit curve in E, meaning that in this graph, the isoprofit curve behaves like the budget set of the consumer. Specifically, the slope of the isoprofit in this scenario is exactly the same of the budget set:

$$\text{Isoprofit } \equiv \bar{\Pi} = p \cdot x_2 - w \cdot z \quad \Leftrightarrow \quad y = \frac{\Pi}{p} - \frac{w}{p} \cdot z.$$

17.2 General Equilibrium and Welfare

We turn now to the welfare analysis. Consider an economy with the following characteristics:

- Consumers: $i = 1, \ldots, \mathcal{I}$;
- Consumption set: $X_i \subset \mathbb{R}^L$, where negative entries in X_i are inputs for firms;
- Preference relation $\succsim_i$ on X_i;
- Firms: $j = 1, \ldots, \mathcal{J}$;
- Technology: $Y_j \subset \mathbb{R}^L$;
- Goods: $l = 1, \ldots, L$;
- Endowment vector $\bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_L)$;
- Economy described by: $\{(x_i, \succsim_i)_{i=1}^{\mathcal{I}},\ \{Y_j\}_{j=1}^{J},\ \bar{\omega}\}$.

Def.[27]: let the allocation $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_{\mathcal{I}}, y_j, \ldots, y_{\mathcal{J}})$ be a specification of a consumption vector $\mathbf{x}_i \in X_i$, $\forall i$, and a production vector $\mathbf{y}_j \in Y_j$, $\forall j$. We say that an allocation is **feasible** if:

$$\sum_i x_{li} = \bar{\omega}_l + \sum_j y_{lj}, \ \forall \text{ good } l$$

or

$$\sum_i x_i = \bar{\omega} + \sum_j y_j.$$

Moreover, we denote the set of feasible allocations as:

$$A = \{(\mathbf{x}, \mathbf{y}) \in X_1 \times \cdots \times X_{\mathcal{I}} \times Y_1 \times \cdots \times Y_{\mathcal{J}} : \sum_i x_i = \bar{\omega} + \sum_j y_j\} \subset \mathbb{R}^{L(\mathcal{I}+\mathcal{J})}.$$

[27]This definition can be found in MWG, p. 546, 16.B.1.

The idea is simple: an allocation is feasible if, and only if, it can be purchased by everyone in the economy with the available income. Actually, the equality condition is even stronger: a feasible allocation exhausts all income from all individuals in the economy.

Def.[28]: a feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto optimal if there's no other allocation $(\mathbf{x}', \mathbf{y}')$ such that $\mathbf{x}'_i \succsim_i \mathbf{x}_i$, $\forall i$, and $\mathbf{x}'_i \succ_i \mathbf{x}_i$ for some i.

This means that $(\mathbf{x}, \mathbf{y})$ is PO if, and only if, (a) it is at least as good as any other feasible allocation from the perspective of all consumers i, and (b) it is not strictly worse than any other allocation for some consumer i. The first part is straightforward, it's like a restatement of the definition of Pareto efficiency. The second part is telling us that an allocation is only PO if there's no one who claims "this allocation is strictly worse than another one for me".

17.2.1 Private Economy

Private economies are characterized by the following:

- Consumers and firms take prices as given (competitive economy);
- Consumers **own** the firms, and have shares on profits;
- Consumer i:
  - ▷ endowment $\omega_i \in \mathbb{R}^L$ $(\bar{\omega} = \sum_i \omega_i)$;
  - ▷ share $\theta_{ij} \in [0, 1]$ on profits of j, $\forall j$, such that
  $$\sum_i \theta_{ij} = 1, \forall j.$$
- Economy described by: $(\{X_i, \succsim_i\}_{i=1}^{\mathcal{I}}, \ Y_j{}_{j=1}^{\mathcal{J}}, \ \{(\omega_i, \theta_{i1}, \ldots, \theta_{i\mathcal{J}})\}_{i=1}^{\mathcal{I}})^{(*)}$.

Def.[29]: Given a private economy characterized by $(*)$, an allocation $(\mathbf{x}^*, \mathbf{y}^*)$, and a price vector $\mathbf{p} = (p_1, \ldots, p_L)$, we say that this allocation is a Walrasian Equilibrium if:

(i) $\forall j$, $\mathbf{y}_j^*$ max profits on Y_j: $\mathbf{p}'\mathbf{y}_j \leq \mathbf{p}'\mathbf{y}_j^*$, $\forall \mathbf{y}_j \in Y_j$

(ii) $\forall i$, $\mathbf{x}_i^*$ max utility representing $\succsim_i$ in the budget set: $\{\mathbf{x}_i \in X_i : \mathbf{p}'\mathbf{x}_i \leq \mathbf{p}'\omega_i + \sum_j \theta_{ij} \cdot \mathbf{p}'\mathbf{y}_j^*\}$

(iii) $\sum_i \mathbf{x}_i^* = \bar{\omega} + \sum_j \mathbf{y}_j^*$.

Proof. We know that $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{p})$ is a price equilibrium with transfers, which implies that it's locally non-satiated

$$\text{if } \mathbf{x}_i \succ_i \mathbf{x}_i^* \Rightarrow \mathbf{p}'\mathbf{x}_i > \omega_i,$$
$$\text{if } \mathbf{x}_i \succsim_i \mathbf{x}_i^* \Rightarrow \mathbf{p}'\mathbf{x}_i \geq \omega_i.$$

Assume that $\mathbf{x}_i \succsim_i \mathbf{x}_i^*$, $\forall i$, with $\mathbf{x}_i \succ_i \mathbf{x}_i^*$ for some i. Then, we have that

$$\sum_i \mathbf{p}'\mathbf{x}_i > \sum \omega_i = \mathbf{p}'\bar{\omega}_i + \sum_j \mathbf{p}'\mathbf{y}_j^*.$$

[28]This definition can be found in MWG, p. 547, 16.B.2.

[29]This definition can be found in MWG, p. 547, 16.B.3.

Moreover, $\mathbf{y}_j^*$ maximizes profits at prices $\mathbf{p}$. Thus,

$$\mathbf{p}'\bar{\omega} + \sum_j \mathbf{p}'\mathbf{y}_j^* \geq \mathbf{p}'\bar{\omega} + \sum_j \mathbf{p}'\mathbf{y}_j \Rightarrow \sum_i \mathbf{p}'\mathbf{x}_i > \mathbf{p}'\bar{\omega} + \sum_j \mathbf{p}'\mathbf{y}_j.$$

Hence, $(\mathbf{x}, \mathbf{y})$ cannot be feasible. Indeed, $\sum_i \mathbf{x}_i = \bar{\omega} + \sum_j \mathbf{y}_j$ implies that $\sum_i \mathbf{p}'\mathbf{x}_i = \mathbf{p}'\bar{\omega} + \sum_j \mathbf{p}'\mathbf{y}_j$, which contradicts the expression above.

Therefore, $(\mathbf{x}^*, \mathbf{y}^*)$ is PO.

□

Despite the (not very intuitive) proof, this last definition simply says that in a private economy, equilibrium is given by the same features as before – namely, feasible allocations being PO, and allocations leading to maximum profits. The difference is in the θ parameters, which define the share of each j company an individual i possesses. The idea is that income is now given not only by endowments ω, but also by this shares of companies, θ.

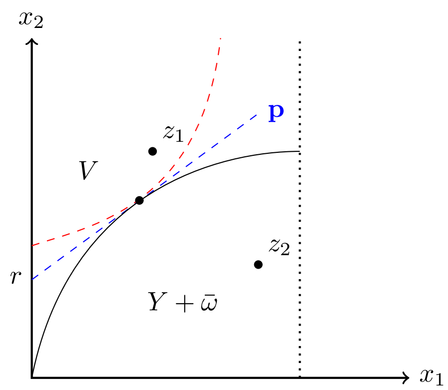
18 Classes 18 & 19

This was a doubled-time class, and prof. Rodrigo used it to provide proofs on the Second Welfare Theorem, and on the existance of a Walrasian Equilibrium in an Exchange Economy. Those proofs are not intuitive at all, and it took about 2 hours to go through them. Prof. Rodrigo used slides to show them, and that's why they will not be exposed here. The only thing I'll make available here are my notes on two of the ten steps of the Second Welfare Theorem's proof, and then I'll move on to a discussion of Pareto efficiency in the context of the Second Welfare theorem.

18.1 Notes on the Proof of the Second Welfare Theorem

18.1.1 Notes on Step 4

Step 4 of the proof of the Second Welfare Theorem comes directly from the Separating Hyperplane Theorem, and we can have a look of it in the figure below.



The idea of r it's just like an income level of a regular budget constraint: $p_1 z_1 + p_2 z_2 = r$. The only thing is that in this formulation, when dealing with general equilibrium, r is actually an aggregate income level that leads to optimal bundles.

What this step is telling us is that there is always a price vector $\mathbf{p} \gg 0$ such that, when multiplied by a given bundle $z_1 \in V$, will greater than or equal to some scalar r, while another bundle $z_2 \in (Y + \bar{\omega})$, when multiplied by the same vector $\mathbf{p}$, will less than or equal to that same scalar r. From this discussion, we see clearly in the figure that the equilibrium point is found in the tangency of the utility function (in red), the production function (in black), and the price ratio (in blue), i.e., the point where what is produced coincides with what maximizes utility for all consumers, at market prices.

18.1.2 Notes on Step 5

The only important note here is that x_i may not be in V, once V is the set of all **strictly** better bundles. The key takeaway is that, if $x_i \succsim_i x_i^*$, $\forall i$, then we can assure that x_i will be in the "frontier" (so to speak) of V.

18.1.3 Notes on Step 7

Consider that there are J firms. If we deviate only one firm from the equilibrium with another level of production, we know that this other production level is feasible, because all other $J-1$ firms (which didn't deviate from the equilibrium) are producing at feasible production levels. [30]

In the end, this step is telling us that all firms are maximizing profits:

$$\forall \mathbf{y}_j \in Y_j,\ \mathbf{y}_j + \sum_{h \neq j} \mathbf{y}_h^* \in Y,$$

where Y_j is the set of all that the j-th firm can produce, and Y is the set of all that all firms can produce at the aggregate level.

18.2 Pareto Efficiency and Social Welfare (MWG, ch. 16, p. 558)

Def.: A social welfare (Bergson-Samuleson) function is a function $W : \mathbb{R}^{\mathcal{I}} \to \mathbb{R}$ that attributes a value for each vector $(\mathbf{u}_1, \ldots, \mathbf{u}_{\mathcal{I}}) \in \mathbb{R}^{\mathcal{I}}$ possible, where $\mathbf{u}_i$ is the utility[31] of the i-th individual:

$$W(\mathbf{u}_1, \ldots, \mathbf{u}_{\mathcal{I}}).$$

Utility Possibility Set:

$$U = \{(\mathbf{u}_1, \ldots, \mathbf{u}_{\mathcal{I}}) \in \mathbb{R}^{\mathcal{I}} : \exists \text{ a feasible allocation } (\mathbf{x}, \mathbf{y}) \text{ s.t. } \mathbf{u}_i \leq \mathbf{u}_i(\mathbf{x}_i),\ i = 1, \ldots, \mathcal{I}\},$$

which means that U is composed of all utility functions such that there is a bundle $\mathbf{x}$ which provides positive utility for all individuals.

Pareto Frontier:

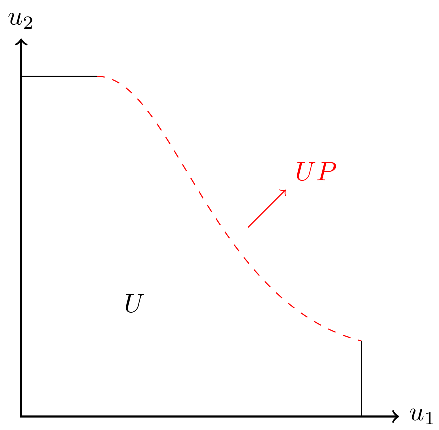
$$UP = \{(\mathbf{u}_1, \ldots, \mathbf{u}_{\mathcal{I}}) \in U : \nexists\ (\mathbf{u}_1', \ldots, \mathbf{u}_{\mathcal{I}}') \in U \text{ s.t. } \mathbf{u}_i' \geq \mathbf{u}_i,\ \forall i \text{ and } \mathbf{u}_i' > \mathbf{u}_i \text{ for some } i\},$$

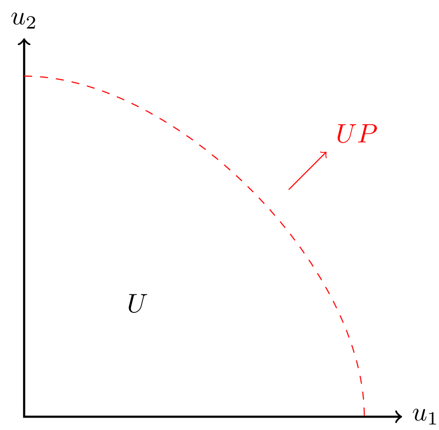
which means that UP is composed of all utility functions $\mathbf{u}_i$ in U such that there is no other utility function (in U) that is at least as good as $\mathbf{u}_i$ for everyone, and at the same time is strictly better than $\mathbf{u}_i$ for someone. It's like an unanimity feature: $\mathbf{u}_i$ is preferred by everyone in the (almost) same way.

The figures below depicts both U and UP sets for a two-consumer economy. Note that only the one at right is a convex set.

[30] There is a formal discussion of this implication, but in the context of the excess demand function (something like "if all markets but the last one are in equilibrium, than the last one will also necessarily be in equilibrium"), which can be found at JR, ch. 5, pp. 204-206 (properties of aggregate excess demand functions).

[31] Recall that the value of a utility function has no intrinsic value, which implies that social welfare functions also doesn't have any economic meaning by themselves.



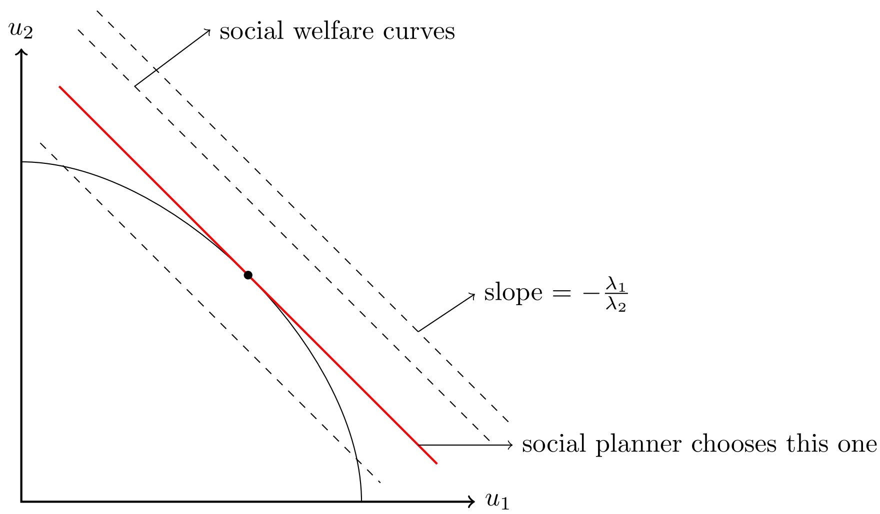


Now, define a linear social welfare function $W(\mathbf{u}_1, \ldots, \mathbf{u}_{\mathcal{I}}) = \sum_i \lambda_i \cdot \mathbf{u}_i$, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{\mathcal{I}})'$ is a vector of constants, and where each $\lambda_i \geq 0$ is like "social weights". Let $\mathbf{u} = W(\mathbf{u}_1, \ldots, \mathbf{u}_{\mathcal{I}})'$ s.t. $W(\mathbf{u}) = \boldsymbol{\lambda}'\mathbf{u}$.

Then, we have that the problem faced by a central planner who wishes to maximize social welfare is given by

$$\max_{\{\omega \in U\}} \boldsymbol{\lambda}'\mathbf{u},$$

which means that the central planner chooses the highest social welfare curve given the utility possibility set.

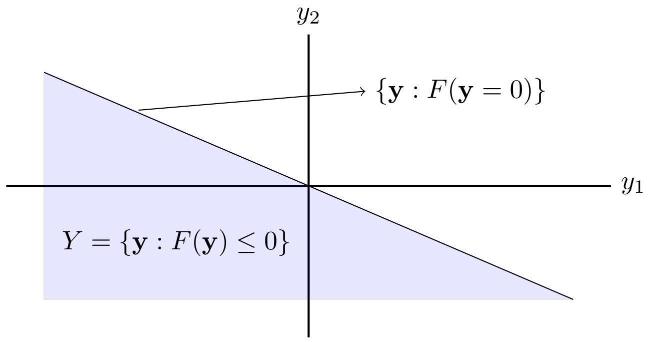


Proposition: If $\mathbf{u}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_{\mathcal{I}}^*)$ is the solution to the central planner's problem above with $\boldsymbol{\lambda} \gg 0$, then $\mathbf{u}^* \in UP$. In other words, $\mathbf{u}^*$ is the utility vector of a PO allocation. Moreover, if U is convex, for any $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_{\mathcal{I}}) \in UP$, there's a set of weights $\boldsymbol{\lambda} \neq 0$ such that $\mathbf{u}$ is the solution to the maximization of some linear social welfare function.

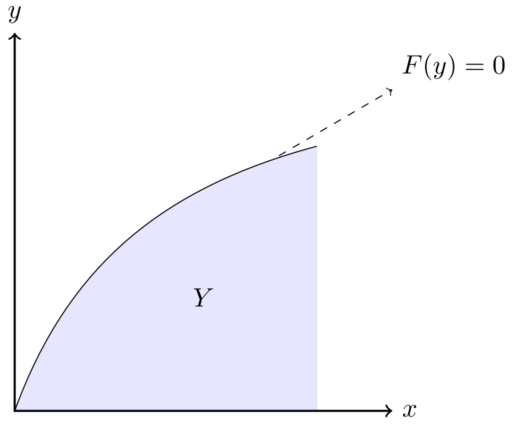
19 Class 20

19.1 First-order Conditions for Pareto Optimality (MWG, ch. 16, p. 561)

- $\mathbf{u}_i(\mathbf{x}_i)$ $\forall i$ is differentiable, strictly monotonic, $\mathbf{u}_i(\mathbf{0}) = 0$, $\forall i$;
- production set of firms is given by: $Y_j = \{\mathbf{y} \in \mathbb{R}^{\mathcal{L}} : F_j(\mathbf{y}) \leq 0\}$, where $F_j(\mathbf{y}) = 0$ is the transformation frontier for firm j.



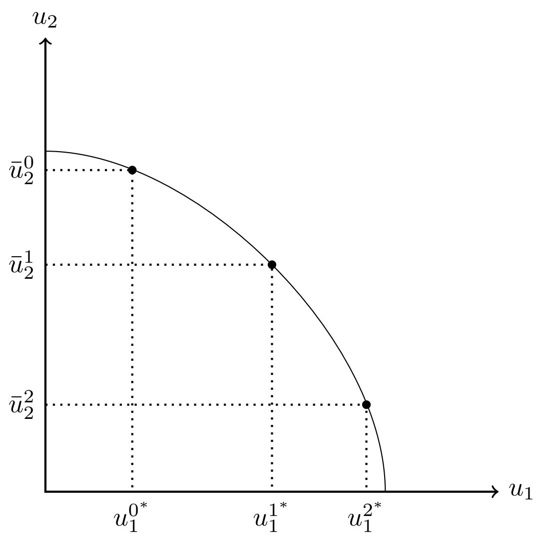
For example, consider the following production function: $y = f(\mathbf{x})$, with y being a scalar. Thus, $F(\mathbf{x}, y) = y - f(\mathbf{x})$.



Pareto optimal allocations can always be characterized as solutions to the following problem (considering just individual 1):

$$\max\ u_1(x_{11}, \ldots, x_{\mathcal{L}1}) \quad s.t. \quad \begin{cases} (1)\ u_i(x_{1i}, \ldots, x_{\mathcal{L}i}) \geq \bar{u}_i,\ i = 2, \ldots, \mathcal{I} \text{ (individuals)}; \\ (2)\ \sum_i x_{li} \leq \bar{\omega}_l + \sum_j y_{lj},\ l = 1, \ldots, \mathcal{L} \text{ (goods)}; \\ (3)\ F_j(y_{1j}, \ldots, y_{\mathcal{L}j}) \leq 0,\ j = 1, \ldots, \mathcal{J} \text{ (firms)}. \end{cases}$$

The following figure depicts the situation parameterizing the utility frontier of the utility possibility set in an economy with only two consumers by required utility level of consumer 2.



Denote by $(\delta_2, \ldots, \delta_{\mathcal{I}}) \geq 0$, $(\mu_1, \ldots, \mu_{\mathcal{L}}) \geq 0$, and $(\gamma_1, \ldots, \gamma_{\mathcal{J}}) \geq 0$ the multipliers associated with the constraints (1), (2), and (3), respectively, and define $\gamma_1 = 1$. Then, the first-order conditions of the max problem proposed is

$$\delta_i \cdot \frac{\partial u_i}{\partial x_{li}} - \mu_l = 0, \ \forall i, l;$$

$$\mu_l - \gamma_j \cdot \frac{\partial F_j}{\partial y_{lj}} = 0, \ \forall j, l.$$

From now on, we're assuming internal solutions. From these FOC's, we get that

$$\underbrace{\frac{\partial u_i / \partial x_{li}}{\partial u_i / \partial x_{l'i}}}_{MRS_i} = \underbrace{\frac{\partial u_{i'} / \partial x_{li'}}{\partial u_{i'} / \partial x_{l'i'}}}_{MRS_{i'}}, \ \forall i, i', l, l';$$

$$\underbrace{\frac{\partial F_j / \partial u_{lj}}{\partial F_j / \partial y_{l'j}}}_{MRTS_j} = \underbrace{\frac{\partial F_{j'} / \partial y_{lj'}}{\partial F_{j'} / \partial y_{l'j'}}}_{MRTS_{j'}}, \ \forall j, j', l, l';$$

$$\underbrace{\frac{\partial u_i / \partial x_{li}}{\partial u_i / \partial x_{l'i}}}_{MRS_i} = \underbrace{\frac{\partial F_j / \partial y_{lj}}{\partial F_j / \partial y_{l'j}}}_{MRTS_j}, \ \forall i, j, l, l'.$$

This last equation is telling us that what firms "trade/exchange" between 2 goods must be equal to the "trade/exchange" that consumers do of these same 2 goods.

19.2 Walrasian Equilibrium (MWG, ch. 16, p. 565)

We proceed now to the relations between the first-order conditions we've just seen with the welfare theorems.

$$\max_{\mathbf{x}_i} \ u(\mathbf{x}_i) \quad \text{s.t.} \quad \mathbf{p}'\mathbf{x}_i \leq \boldsymbol{\omega}_i \quad (\alpha_i)$$

$$\max_{\mathbf{y}_j} \ \mathbf{p}'\mathbf{y}_j \quad \text{s.t.} \quad F_j(\mathbf{y}_j) \leq 0 \quad (\beta_j)$$

The first-order conditions are

$$\frac{\partial u_i}{\partial x_{li}} = \alpha_i \cdot p_l, \quad \forall i, l;$$

$$p_l = \beta_j \cdot \frac{\partial F_j}{\partial y_{lj}}, \quad \forall j, l.$$

If $\mu_l = p_l$, $\delta_i = 1/\alpha_i$, and $\gamma_j = \beta_j$, then these FOC's are identical to the ones of the PO problem.

Now, considering all individuals, we have that the maximum of the social welfare function is given by

$$\max \sum_i \lambda_i \cdot u(\mathbf{x}_i) \quad \text{s.t.} \quad \begin{cases} \sum_i x_{li} \leq \bar{\omega}_l + \sum_j y_{lj}, & \forall l \quad (\psi_l); \\ F_j(\mathbf{y}_j) \leq 0, & \forall j \quad (\eta_j). \end{cases}$$

Here, the first-order conditions are

$$\lambda_i \cdot \frac{\partial u(\mathbf{x}_i)}{\partial x_{li}} = \psi_l \quad \forall i, l;$$

$$\psi_l = \eta_j \cdot \frac{\partial F_j}{\partial y_{lj}}, \quad \forall j, l.$$

If $\delta_i = \frac{\lambda_i}{\lambda_1}$, $\mu_l = \frac{\psi_l}{\lambda_1}$, and $\gamma_j = \frac{\eta_j}{\lambda_1}$, then this problem also leads to the same solution of the initial PO problem. For the Walrasian Equilibrium, we then have that

$$\alpha_i = \frac{l}{\lambda_i}, \quad p_l = \psi_l, \quad \text{and} \quad \beta_j = \eta_j.$$

19.3 Externalities (MWG, ch. 11, p. 350)

"Externalities" and "Public Goods" are the latest subjects of the course. The basic reference is MWG, and this part of the course was really fast. Prof. Rodrigo notes on it are brief, such that there is no definition/motivation of the externality problem. For more details on this, check MWG, ch. 11.

We begin considering the classical case of a productive externality:

- 2 firms (or firm and individual);
  - ▷ $\Pi_1 = px - c(x)$;
  - ▷ $\Pi_2 = -e(x)$;
  - ▷ $e', e'', c', c'' > 0$.
- In a decentralized solution, firm 1 chooses (first-order condition): $p = c'(x_1)$;
- Total social surplus:

$$\max_x \; px - c(x) - e(x) \Rightarrow p = c'(x_s) + e'(x_s).$$

  Since $c'' > 0$, and $e' > 0$, we have that $x_s < x_1$, i.e., firm 1 produces more than the socially optimum, onde it doesn't consider the externality imposed on firm 2. This also means that

$$\text{Private } MgC < \text{Social } MgC.$$

20 Class 21

20.1 Externalities (cont'd)

- Common solution to equalize $PMgC$ and $SMgC$: Pigouvian Taxation.

20.1.1 Pigouvian Taxation

Tax t on good x. Profits then become $\Pi_1 = (p - t)x - c(x)$:

$$p = c'(x) + t.$$

By making $t^* = e'(x_s)$, i.e., tax as the marginal value of the externality evaluated at the social optimum, would make:

$$p = c'(x_s) + t^* = c'(x_s) + e'(x_s).$$

But if the government knows all of it, then it could also just impose quotas on production, in the sense that this quota would restrict firm's production to x_s.

There's a surging discussion: externalities may be connected to the absence of a market for trading them.[32] This brings us to the following:

- Missing markets (undefined property rights): Assume there's a (competitive) market for emissions between firms 1 and 2. Price r is the one that firm 1 has to pay to firm 2 (2 owns property rights). Then,

$$\Pi_1 = (p - r)x - c(x) \quad \xrightarrow{\text{FOC's}} \quad p - r = c'(x) \Rightarrow r = p - c'(x);$$
$$\Pi_2 = rx - e(x) \quad \xrightarrow{\text{FOC's}} \quad r = e'(x);$$
$$\text{equilibrium } r^* \Rightarrow p = c'(x_s) + e'(x_s).$$

[32] Another possible solution for externalities, when they're well contained, is to integrate/merge both firms, the one harmed and the one harming.

In this example, there's a market force towards integration: both firms working together have higher profit than separately.

Coase's Theorem: "If property rights are well-defined, and there are no transaction costs nor income effects, then the solution is Pareto optimal, and the final allocation doesn't depend on the assignment of property rights."

- With income effects, the solution is still efficient, but the final allocation may change depending on the assignment of property rights;
- Example: neighbor A values \$2 to listen to loud music, while neighbor B values \$4 to silence. Then,

  ▷ Property right to B: final allocation has no music, and no payment (received);

  ▷ Property right to A: final allocation has no music, and some payment between \$2 and \$4 from B to A, once B values silence more than what A values listening to loud music.

20.2 Public Goods (MWG, ch. 11, p. 350)

By definition, a public good is a good that is non-excludable and non-rival. Non-excludable means that it is impossible for one to effectively avoid other people to enjoy the good, i.e., there is no (effective) way to tax people for the use of that good, or to prohibit someone to have access to it. Non-rival means that the use/consumption of the good by someone does not restrain another one to also use/consume that good.

By these reasons, these goods are called "public", in the sense that, in general, the state is the provider of those goods. The key question is then: what is the efficient level of provision of a public good? We begin considering the following characterization of a given economy:

- $G = f(\sum_i g_i)$;
- $u_i(G, x_i)$;
- Endowment ω_i can be used for x_i or g_i;
- Prices are all $= 1$;
- PO solution (considering δ as the multiplier):

$$\sum_i \lambda_i \cdot u_i(G, x_i) \quad \text{s.t.} \quad \sum \omega_i = \sum_i g_i + \sum_i x_i$$

$$\begin{aligned}
\textbf{FOC's} \quad (x_i): \ & \lambda_i \cdot \frac{\partial u_i}{\partial x_i} = \delta, \quad \forall i \\
(g_i): \ & \sum_j \lambda_j \cdot \frac{\partial u_j}{\partial G} \cdot \frac{\partial G}{\partial g_i} = \delta, \quad \forall i \\
& \sum_j \lambda_j \cdot \frac{\partial j}{\partial G} \cdot f' = \delta, \quad \forall i \\
\Rightarrow & \sum_j \lambda_j \cdot \frac{\partial u_j}{\partial G} \cdot f' = \lambda_i \cdot \frac{\partial u_i}{\partial x_i}, \quad \forall i \\
\text{Also,} \quad & \lambda_j \cdot \frac{\partial u_j}{\partial x_j} = \lambda_i \cdot \frac{\partial u_i}{\partial x_i}, \quad \forall i, j \\
& \sum_j \frac{\partial u_j / \partial G}{\partial u_j / \partial x_j} \cdot f' = 1 \ \Leftrightarrow \ \sum_j \frac{\partial u_j / \partial G}{\partial u_j / \partial x_j} = \frac{1}{f'}
\end{aligned}$$

That's the general characterization of the provision of a public good:

$$\sum_j MRS^j_{Gx} = MRTS_{Gx}.$$

Now, turning to the private provision of a public good, we have the following characterization:

- Individual takes g_j's for $j \neq i$ as given. Thus,

  $$\max_{g_i} \; u_i(G, \omega_i - g_i) \quad \text{s.t.} \quad G = f(\sum_j g_j),$$

  where $g_j \neq g_i$ is taken as given. The first-order condition is

  $$\frac{\partial u_i}{\partial G} \cdot f' - \frac{\partial u_i}{\partial x_i} = 0$$

  $$\Rightarrow \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i} = \frac{1}{f'} \; \Leftrightarrow \; MRS^i_{Gx} = MRTS_{Gx}.$$

  Notice that there may be a corner solution if

  $$\text{If} \quad \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i} < \frac{1}{f'} \text{ at } g_i = 0 \Rightarrow g_i^* = 0.$$

- Equilibrium: $G^* = f(\sum_i g_i^*)$, $(x_1^*, \ldots, x_{\mathcal{I}}^*)$, such that (g_i^*, x_i^*) max utility for every i taking g_j^* for $i = j$ as given;

- Inefficiently low provision of G^*. Possible solution: Lindahl Pricing.

Lindahl Pricing. We have the following max problem for private provision of a public good:

$$\max_{\{x_i, G\}} \; u_i(G, x_i) \quad \text{s.t.} \quad x_i + p_i G = \omega_i,$$

where we notice that each i individual is charged with different prices p_i for the provision of the public good G. The first-order condition in this case is

$$\frac{\partial u_i/\partial G}{\partial u_i/\partial x_i} + \sum_{j \neq i} \frac{\partial u_j/\partial G}{\partial u_j/\partial x_j} = \frac{1}{f'} \Rightarrow \frac{\partial u_i/\partial G}{\partial u_i/\partial x_i} = p_i \Rightarrow p_i = -\sum_{j \neq i} \frac{\partial u^e_j/\partial G}{\partial u^e_j/\partial x_j} + \frac{1}{f'}^{e} = \frac{\partial u^e_i/\partial G}{\partial u^e_i/\partial x_i}.$$

The idea of Lindahl pricing is charging more (less) who likes more (less) the public good.

21 Class 22

This was the final class of the course, dedicated to discuss some points raised by the students, and for a brief discussion about the Arrow's impossibility Theorem. For further details, check the references.

21.1 Social Welfare Functions and Arrow's Impossibility Theorem

Consider the following characterization:

- X: set of "social tastes";
- $N \geq 2$ individuals;
  - ▷ preferences: R^i over X; p^i strict preferences; I^i indifference.
- "Social Preference" relation R defined based on R^i's:

  $$R = f(R^1, \ldots, R^I).$$

- Example: Condorcet Paradox.

  ▷ $N = 3$;

  ▷ $X = \{x, y, z\}$;

  ▷ p^i :

1	2	3
x	y	z
y	z	x
z	x	z

  ▷ Social choice rule: majority voting.

$$x\text{P}y,\ y\text{P}z,\ z\text{P}x \quad \Rightarrow \quad \text{violates transitivity.}$$

Arrow's Impossibility Theorem. If there are at least 3 states in X, then generically, there is no social welfare function f that simultaneously satisfies the following conditions:

- Unrestricted Domain: domain of f includes all possible combinations of R^i over X;
- Weak Pareto Principle (unanimity): $\forall x, y \in X$, if $x\text{P}^i y\ \forall i \ \Rightarrow \ x\text{P}y$;
- Independence of Irrelevant Alternatives: consider $R = f(R^1, \ldots, R^N)$ and $R = f(R^1, \ldots, R^N)$. If every i orders x and y the same way under R^i, the social ordering of x and y should be the same under R and R. This condition has this fancy name because, in practice, it rules out the possibility of "useful vote" ("*voto útil*") in a context of elections;
- Non-dictatorship: $\nexists\, i$ s.t. $\forall x, y \in X,\ x\text{P}^i y \ \Rightarrow \ x\text{P}y$, regardless of R^j for $j \neq i$.

22 Exercises (Diverse)

22.1 TR, p. 67, exercise 1.1

22.1.1 Item (i) - Find total welfare

We have that $W^c \equiv$ consumer surplus + producer profit.

$$W^c = \max_p \left(\underbrace{\int_p^\infty x^{-\varepsilon}\, dx}_{CS} + \underbrace{(p - c) \cdot p^{-\varepsilon}}_{\text{profit}} \right).$$

Here, we have that the consumer surplus is the integral of its demand function from price p (minimum possible, market price), up to ∞, considering that the monopolist could charge an arbitrarily high price. We know from previous notions that CS will be maximized when prices are the lowest possible, i.e., prices of perfect competition, $p = MgC = c$. Thus,

$$W^c = \max_p \left(\int_p^\infty x^{-\varepsilon}\, dx + (c - c) \cdot p^{-\varepsilon} \right) = \max_p \left(\int_p^\infty x^{-\varepsilon}\, dx \right).$$

Solving it, we get that

$$\int_p^\infty x^{-\varepsilon}\, dx = \int_p^\infty \frac{1}{x^\varepsilon}\, dx = -\frac{1}{(\varepsilon - 1) \cdot x^{\varepsilon - 1}} \Bigg|_c^\infty = -\frac{x^{1-\varepsilon}}{(\varepsilon - 1)} \Bigg|_c^\infty$$

$$= -\underbrace{\frac{\infty^{1-\varepsilon}}{(\varepsilon - 1)}}_{=0} - \left(-\frac{c^{1-\varepsilon}}{(\varepsilon - 1)} \right) = \frac{c^{1-\varepsilon}}{(\varepsilon - 1)}$$

22.1.2 Item (ii) - Compute WL under monopoly

Welfare under monopoly, W^m, corresponds to price $p^m = \frac{c}{1-\frac{1}{\varepsilon}}$. This comes from the alternative version of the first-order condition of the monopolist's profit max problem:

$$\underbrace{p'(q) \cdot q + p(q)}_{MgR} = \underbrace{c'(q)}_{MgC=c} .$$

Note that

$$MgR = \frac{\partial p}{\partial q} \cdot q + p = p \cdot \left(\frac{\partial p}{\partial q} \cdot \frac{q}{p} + 1 \right) = p \cdot \left(-\frac{1}{\varepsilon} + 1 \right) = p \cdot \left(1 - \frac{1}{\varepsilon} \right).$$

Thus,

$$p^m = \frac{c}{1-\frac{1}{\varepsilon}}.$$

Hence, the welfare loss is $W^c - W^m$, i.e., what consumers lose of surplus under monopoly. Thus,

$$\begin{aligned} WL &= W^c - W^m \\ &= \frac{c^{1-\varepsilon}}{\varepsilon - 1} - \frac{c}{1-\frac{1}{\varepsilon}} \\ &= \frac{c^{1-\varepsilon}}{\varepsilon - 1} - \frac{c \cdot \varepsilon}{\varepsilon - 1} \\ &= \frac{c^{1-\varepsilon} - c\varepsilon}{\varepsilon - 1} \\ &= \frac{c \cdot c^{-\varepsilon} - c \cdot \varepsilon}{\varepsilon - 1} \\ &= \frac{c(c^{-\varepsilon} \cdot \varepsilon)}{\varepsilon - 1} \\ &= \frac{c(\varepsilon / c^{\varepsilon})}{\varepsilon - 1} > 0 \end{aligned}$$

because $c > 0$, $\varepsilon / c^{\varepsilon} > 0$, and $\varepsilon - 1 > 0$.

22.1.3 Item (iii) - Properties of WL

$$WL = \left(\frac{c^{1-\varepsilon}}{\varepsilon - 1} \right) \cdot \left[1 - \left(\frac{2\varepsilon - 1}{\varepsilon - 1} \right) \cdot \left(\frac{\varepsilon}{\varepsilon - 1} \right)^{-\varepsilon} \right] > 0$$

$$\frac{WL}{W^c} = \frac{WL}{c^{1-\varepsilon}/(\varepsilon - 1)} = \left[1 - \left(\frac{2\varepsilon - 1}{\varepsilon - 1} \right) \cdot \left(\frac{\varepsilon}{\varepsilon - 1} \right)^{-\varepsilon} \right] = 1 - k(\varepsilon),$$

where $ln(k(\varepsilon)) = ln(2\varepsilon - 1) - ln(\varepsilon - 1) - \varepsilon ln(\varepsilon) + \varepsilon ln(\varepsilon - 1)$. Thus,

$$\begin{aligned} \frac{k'(\varepsilon)}{k(\varepsilon)} &= \frac{2}{2\varepsilon - 1} - \frac{1}{\varepsilon - 1} - 1 - ln\varepsilon + \frac{\varepsilon}{\varepsilon - 1} + ln(\varepsilon - 1) \\ &= \frac{2}{2\varepsilon - 1} + \underbrace{ln\left(\frac{\varepsilon - 1}{\varepsilon} \right)}_{\text{negative}}, \end{aligned}$$

which means that $1 - k(\varepsilon)$ increases with ε.

Further, we note that the monopolist maximizes $p \cdot q(p) - c(q(p))$, which leads to the following first-order condition

$$q'(p^m) \cdot p^m + q(p^m) - c'(q(p^m)) \cdot q'(p^m) = 0$$

$$q(p^m) + q'(p^m)[p^m - c'(q(p^m))] = 0$$

$$p^m - c'(q(p^m)) = -\frac{q(p^m)}{q'(p^m)}$$

$$\frac{p^m - c'(q(p^m))}{p^m} = -\frac{q(p^m)}{q'(p^m) \cdot p^m} = \frac{1}{\varepsilon}.$$

Note that if $c'(q(p^m)) > 0$, then

$$\frac{p^m - c'(q(p^m))}{p^m} < \frac{p^m - 0}{p^m} = 1.$$

Thus, $\frac{1}{\varepsilon} < 1 \Leftrightarrow \varepsilon > 1$, i.e., monopolists always operates on the elastic portion of the demand curve $q(p)$.

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